On distinguishing proof-theoretic consequence from derivability

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According to the common conception of *logical consequence*, it can be defined in two main ways:
- **Model-theoretically:** For a suitable notion of a model, consequence is taken to be *preservation* (also called *propagation or transmission*) of truth over all models.

$$\Gamma \models \varphi \text{ iff for every model } \mathcal{M}, \text{if } \mathcal{M} \models \Gamma \text{ then } \mathcal{M} \models \varphi \tag{1}$$

- **Proof-Theoretically:** For a suitable complete proof-system $\mathcal{N}$, consequence is taken as *derivability* in $\mathcal{N}$, denoted '$\vdash_{\mathcal{N}}$'.

$$\Gamma \vdash_{\mathcal{N}} \varphi \text{ iff there exists a derivation } \mathcal{D} \text{ of } \varphi \text{ from (open) assumptions } \Gamma \tag{2}$$

The idea that logical consequence involves preservation of *something*, not necessarily of truth, has been suggested by many. Some examples:
- *Information:* Propagation of the information (in a situation) is underlying consequence of Relevant Logic.
- *Ambiguity:* a proposition p is taken as *ambiguously* between two different propositions, p_t and p_f. A *measure* of ambiguity of an inconsistent Γ is defined as the minimal number of proposition in Γ the treatment of which as ambiguous renders Γ consistent. Propagation of ambiguity is used for defining consequence for paraconsistent logics.
- *Precisification:* In the context of *vagueness*, Logical consequence is preservation of *super-truth* (i.e., truth in all *precisifications*).
A natural question arising is, what is common to all the "things" being suggested as preserved, or propagated, by the various consequence relations mentioned above?
I want to argue that they all serve (either explicitly or implicitly) as *central concepts on which theories of meaning are based.*
Two of the main theories of meaning are the following.
– In **Model-Theoretic Semantics** (MTS), The central concept is *truth* (in arbitrary models). Meaning is defined as truth-conditions.
– In **Proof-Theoretic Semantics** (PTS), The central concept is *proof*, or more generally, *canonical derivation*, in appropriate meaning-conferring proof-systems.:
Meaning is *determined* (either implicitly or, as I prefer, explicitly) by the rules of the meaning-conferring system.
– The other propagated "things" mentioned above have a similar role as theories of meaning for Relevant Logic, general paraconsistent logics and for languages with vagueness.
Consequently, I suggest the following informal principle:
meaning-based logical consequence: In a theory of meaning $\mathcal{T}$, logical consequence is based on the propagation of the central concept of $\mathcal{T}$.
In this paper I argue that, in spite of the coextensiveness in many logics of derivability and preservation of truth in models, if one adheres to the proof-theoretic semantics theory of meaning then (2) is not the right definition of proof-theoretic consequence. While (1) is faithful to the usual model-theoretic conception of meaning, (2) is not faithful to the PTS conception of meaning.

Instead, I suggest the following conception of proof-theoretic consequence.
– The *(proof-theoretic) meaning* $[\![\varphi]\!]$ of φ, is given by: $[\![\varphi]\!] = \lambda\Gamma.[\![\varphi]\!]^c_\Gamma$, a function from contexts Γ to the (possibly empty) collection of *canonical derivations* of φ from Γ ($\Gamma \vdash^c \varphi$).
– The definition of *proof-theoretic consequence* (pt-consequence) rests on the notion of *grounds for assertion* for φ, closely related to $[\![\varphi]\!]$, given by: $GA[\![\varphi]\!] = \{\Gamma \mid \Gamma \vdash^c \varphi\}$.
– Proof-theoretic consequence: ψ is a *proof-theoretic consequence* of Γ ($\Gamma \Vdash \psi$) iff $GA[\![\Gamma]\!] \subseteq GA[\![\psi]\!]$.
The paper studies two definitions of $GA[\![\Gamma]\!]$, based on conjunction (additive) and on fusion (multiplicative).
I show that for intuitionistic logic, but not for classical logic, proof-theoretic consequence coincides with derivability.