Monoidal logics:
De Morgan negations and classical systems

Monoidal logics were introduced in the work of Peterson (2014a,b, 2015) to analyze the proof theory of deontic logic. They are inspired by Lambek's logical understanding of category theory (cf. 1968, 1969 and Lambek and Scott 1986) and by Lawvere's (1963) insight that logical systems can be defined as pairs of adjoint functors. In contrast with Lambek's work, monoidal logics assume category theory as a foundation. Basically, the idea is to define logical systems using specific rules and axiom schemata in order to make explicit the categorical (monoidal) structure of the logics. It starts from the definition of a *deductive system*, defined as a collection of formulas together with a collection of equivalence classes of proofs (i.e., deductions $\varphi \longrightarrow \psi$) satisfying (1) and (cut) below. In a nutshell, the idea is to define different deductive systems in such a way that it can easily be proven that i) they are instances of specific monoidal categories and, further, that ii) their logical connectives are functors with specific properties.

Monoidal logics can be compared to substructural logics (cf. Restall 2000) and, more generally, to display logics (cf. Goré 1998). Indeed, it is possible to define a translation t from the language of monoidal logics to the language of display logics such that a proof $\varphi \longrightarrow \psi$ is derivable within specific monoidal deductive systems if and only if $t(\varphi) \vdash t(\psi)$ is derivable within their respective display counterparts. One upshot of this comparison is that monoidal logics can be proven to be weaker and more flexible than substructural logics. In substructural logics, the elimination of double negation(s) is generally thought to be accompanied by the satisfaction of the de Morgan dualities (cf. Restall 2000, pp.62-5). However, the elimination of double negation(s) can be proven to be independent from the de Morgan dualities in monoidal logics.

This result can be understood in light of what might be defined as a *classical deductive system*. Let $\mathcal{L} = \{Prop, (,), \otimes, 1, \multimap, \triangleright, 0, \oplus, *\}$ (with $Prop$ is a collection of atomic propositions p_i and well-formed formulas defined recursively as usual). Negations are defined by $\sim \varphi =_{df} \varphi \multimap *$ and $\neg\varphi =_{df} \varphi \triangleright *$. A *monoidal closed deductive system with co-tensor* **MC**co**M** is a deductive system satisfying the following rules of inference ($\bullet/i$ stands for either $\otimes/1$ or $\oplus/0$).

$$\frac{}{\varphi \longrightarrow \varphi}\ (1) \qquad \frac{\varphi \longrightarrow \psi \qquad \psi \longrightarrow \rho}{\varphi \longrightarrow \rho}\ (\text{cut}) \qquad \frac{\varphi \longrightarrow \psi \qquad \rho \longrightarrow \tau}{\varphi \bullet \rho \longrightarrow \psi \bullet \tau}\ (\text{t}) \qquad \frac{\varphi \longrightarrow \psi \bullet i}{\overline{\varphi \longrightarrow \psi}}\ (\text{r}) \qquad \frac{\varphi \longrightarrow i \bullet \psi}{\overline{\varphi \longrightarrow \psi}}\ (\text{l})$$

$$\frac{\tau \longrightarrow (\varphi \bullet \psi) \bullet \rho}{\overline{\tau \longrightarrow \varphi \bullet (\psi \bullet \rho)}}\ (\text{a}) \qquad \frac{\varphi \otimes \psi \longrightarrow \rho}{\overline{\varphi \longrightarrow \psi \multimap \rho}}\ (\text{cl}) \qquad \frac{\varphi \otimes \psi \longrightarrow \rho}{\overline{\psi \longrightarrow \varphi \triangleright \rho}}\ (\text{cl'})$$

A **MC**co**M** is *classical* if and only if it satisfies ($\oplus$1) and ($\oplus$2).

$$\varphi \oplus \psi \longrightarrow \neg\psi \multimap \varphi \qquad\qquad \varphi \oplus \psi \longrightarrow \sim \varphi \triangleright \psi \qquad (\oplus 1)$$

$$\neg\psi \multimap \varphi \longrightarrow \varphi \oplus \psi \qquad\qquad \sim \varphi \triangleright \psi \longrightarrow \varphi \oplus \psi \qquad (\oplus 2)$$

A de Morgan negation is usually conceived as a single negation satisfying the elimination of double negation and the de Morgan dualities. Nonetheless, this notion can be generalized to monoidal logics and, as such, *de Morgan negations* can be defined as negations satisfying the elimination of double negations and the de Morgan dualities (**dm1**)-(**dm4**).

$$\neg\psi \otimes \neg\varphi \longrightarrow \neg(\varphi \oplus \psi) \qquad\qquad \sim \psi \otimes \sim \varphi \longrightarrow \sim (\varphi \oplus \psi) \qquad (\textbf{dm1})$$

$$\neg(\varphi \oplus \psi) \longrightarrow \neg\psi \otimes \neg\varphi \qquad\qquad \sim (\varphi \oplus \psi) \longrightarrow \sim \psi \otimes \sim \varphi \qquad (\textbf{dm2})$$

$$\neg\varphi \oplus \neg\psi \longrightarrow \neg(\psi \otimes \varphi) \qquad\qquad \sim \varphi \oplus \sim \psi \longrightarrow \sim (\psi \otimes \varphi) \qquad (\textbf{dm3})$$

$$\neg(\psi \otimes \varphi) \longrightarrow \neg\varphi \oplus \neg\psi \qquad\qquad \sim (\psi \otimes \varphi) \longrightarrow \sim \varphi \oplus \sim \psi \qquad (\textbf{dm4})$$

As it happens, classical deductive systems corresponds precisely to deductive systems with de Morgan negations. Likewise, it corresponds precisely to deductive systems satisfying weak distributivity (a.k.a. linear distributivity) and the law of excluded middle. However, it can be proven that there are **MC**co**M**s satisfying the elimination of double negations but that are neither weakly distributive nor classical and, incidentally, that do not satisfy the de Morgan dualities. From a categorical perspective, classical deductive systems corresponds to deductive systems in which $\oplus$ is a right adjoint functor to $\otimes$. In such a case, there is a Galois connection between the tensor and the co-tensor. This enables the inter-definition of all the logical connectives. Our analysis show that such a connection is not a necessary property of monoidal deductive systems and that there is nothing in the rules governing the behavior of the connectives that implies any special relationship between $\otimes$ and $\oplus$ such as weak distributivity, the de Morgan dualties, ($\oplus$1) or ($\oplus$2).

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