Some Notes on Intuitionistic Logic and Dialogue Semantics

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Preface

These are the lecture notes of an Erasmus+ mini-course given at the Philosophical Department of the University of Belgrade in April 2018. It was part of a course on the philosophy of mathematics held by Miloš Adžić. The purpose of the mini-course was to expand on the philosophical understanding of intuitionism by presenting some central results on intuitionistic logic and its relation to classical logic. Besides proof-theoretic results and Kripke-semantics also dialogue semantics were presented.
I would like to thank all the students for their participation, and I would like to thank Miloš Adžić heartily for his kind invitation and his great hospitality.

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Contents

1 Intuitionistic logic

In intuitionistic logic one investigates principles of deductive reasoning that are based on a constructivistic approach to mathematics which goes back to L. E. J. Brouwer (1881–1966) and A. Heyting (1898–1980), among others. This form of constructivism is called *intuitionism*. Essential aspects are: *intuitionism*

(i) Mental constructions are primary in mathematics. It is not about formal operations with symbols of a language of mathematics. The latter is just an auxiliary means to communicate our mental constructions.

(ii) The view that mathematical statements are true or false independently of our knowledge about them is regarded as being without meaning. A mathematical statement is true, if we have a proof of it; it is false, if we can show that the assumption that there is a proof of the statement leads to a contradiction. Thus we cannot claim for arbitrary statements that they are true or false. Consequently, *tertium non datur* $A \vee \neg A$ does not hold in general; it can hold at best only for finite domains.

(iii) Intuitionism is an opposite standpoint to platonism: In mathematics one does not discover truths about mathematical objects existing independently from us; these objects are rather created by us. It is also possible to investigate constructions that do not terminate.

In the following, we present some results on intuitionistic logic, where we presuppose that the meaning of the logical constants is given along the lines of the Brouwer–Heyting–Kolmogorov interpretation (1.2). Using proof-theoretic methods (1.3) we then discuss the two notions of derivable and admissible rules (1.4 and 1.5), and we show some results on the relation between intuitionistic and classical logic (1.6). Afterwards we introduce Kripke-semantics for intuitionistic logic (1.7). Throughout, we will restrict ourselves to the propositional fragment of intuitionistic logic.

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1.1 Weak counterexamples

To begin with we consider two examples that illustrate why certain laws of classical logic have to be rejected when a constructivistic understanding of the logical constants is presupposed.

Example. We consider the statement

> *There are two irrational numbers x and y, such that x^y is rational.*

It can be proved easily by arguing classically as follows.
$\sqrt{2}$ is irrational, and by *tertium non datur* we have: $\sqrt{2}^{\sqrt{2}}$ is rational or it is not rational, i.e. irrational. We consider both cases:

(i) Assume $\sqrt{2}^{\sqrt{2}}$ is rational. We let $x = \sqrt{2}$ and $y = \sqrt{2}$, such that $x^y = \sqrt{2}^{\sqrt{2}}$, which is a rational number by assumption.

(ii) Assume $\sqrt{2}^{\sqrt{2}}$ is irrational. We let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. Then $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = (\sqrt{2})^2 = 2$, which is rational.

However, the proof given in the example is not a *constructive* proof, since we cannot present two numbers x and y, such that x^y is rational. Under a constructivistic understanding of the considered existential statement, where the existential quantifier is interpreted as "it can be constructed", we have thus not given a satisfying proof of that statement.

Example. We consider the conjecture p

> *There are infinitely many twin primes, i.e. prime numbers n, such that also $n + 2$ is a prime number.*

This conjecture has not been decided yet. That is, we neither have a proof of p nor do we have a proof of $\neg p$. We therefore cannot claim that $p \vee \neg p$ holds.

This is a so-called *weak counterexample* for *tertium non datur*. From the constructivistic point of view, *tertium non datur* $A \vee \neg A$ says that for any statement A we have a proof of A or a proof of $\neg A$, i.e. a construction which transforms a hypothetical proof of A into a proof of the absurdity $\perp$. But then we would be in the position to decide for any statement whether it holds or not. But an example like the statement "There are infinitely many twin primes", whose validity has not been decided yet, shows that this is not the case.
This is only a *weak* counterexample, since *tertium non datur* has not been refuted, i.e. the assumption of *tertium non datur* has not been shown to lead to absurdity. It has only been shown that *tertium non datur* is not an acceptable logical principle from the constructivistic point of view.
Moreover, it is impossible (from a constructivistic or intuitionistic point of view) to refute *tertium non datur* by finding some statement A such that $\neg(A \vee \neg A)$ holds, since $\neg\neg(A \vee \neg A)$ holds intuitionistically for all statements A.

1.2 The BHK-interpretation

The meaning of the logical constants $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\perp$ (*falsum*, absurdity) and $\neg$ (negation) shall be explained more precisely by the following *proof interpretation* or *Brouwer–Heyting–Kolmogorov interpretation* (short: *BHK-interpretation*): *BHK-interpretation*

(H1) a is a proof of $A \wedge B$ iff a is a pair $\langle b, c \rangle$, such that b is a proof of A, and c is a proof of B.

(H2) a is a proof of $A \vee B$ iff a is a pair $\langle b, c \rangle$, such that $b \in \{0, 1\}$ and c is a proof of A, if $b = 0$, and c is a proof of B, if $b = 1$.

(H3) a is a proof of $A \to B$ iff a is a construction that transforms any proof b of A into a proof $a(b)$ of B.

(H4) There is no proof a of $\bot$. A proof a of $\neg A$ is a construction that transforms any hypothetical proof b of A into a proof $a(b)$ of $\bot$.

Remarks. (i) The BHK-interpretation of the logical constants is not a well-founded inductive definition of "is a proof of A", since a base clause defining this notion for all atomic formulas is missing. The BHK-interpretation is rather an informal explication of the meaning of the logical constants.

(ii) The notion of construction can be understood more or less broadly. Intended is an understanding as algorithm or computable function.

(iii) It is usually presupposed that a is a proof of a formula A if and only if a is a proof of arbitrary instances of A. Such a requirement is stated explicitly in Heyting's (1971, p. 103) presentation of the BHK-interpretation.

(iv) In clause (H4) the *falsum* $\bot$ is used as a symbol for an arbitrary contradiction. (In the language of arithmetic this could be the statement $0 = 1$, for example.)

Examples. The following formulas are valid under the BHK-interpretation:

(i) $A \to (B \to A)$: We have to find a construction c that transforms a proof a of A into a proof of $B \to A$. For a given proof a of A the construction $c(b) = a$ is what we are looking for; it maps each proof b of B to the proof a of A.

(ii) $(A \wedge B) \to A$: Let $\langle a, b \rangle$ be a proof of $A \wedge B$. Then the construction c, where $c(a, b) = a$ (i.e. the construction that projects to the first of two arguments), transforms the proof of $A \wedge B$ into a proof of A. By clause (H3), c is a proof of $(A \wedge B) \to A$.

(iii) $\bot \to A$: Since $\bot$ has no proof, any function (e.g. the identity $c(a) = a$) can be taken as a construction that transforms a hypothetical proof of $\bot$ into a proof of A. (Note that the domain of such a function is always empty.)

However, $A \vee \neg A$ is not valid: By clause (H2), $A \vee \neg A$ means that we either have a proof of A or a proof of $\neg A$, for any statement A. But then e.g. the twin prime conjecture would be decided, which, however, is not the case. The *tertium non datur* can thus not hold in general.

1.3 The calculus of natural deduction

The BHK-interpretation can be used to justify the inference rules of the calculus NI of natural deduction for intuitionistic logic. As examples we consider the rules for conjunction and implication:

(i) Clause (H1) justifies the conjunction introduction rule, when read from right to left:

$$\frac{\begin{matrix}\mathscr{D}_b & \mathscr{D}_c \\ A & B\end{matrix}}{A \wedge B}$$

When read from left to right, (H1) justifies the pair of conjunction elimination rules:

$$\frac{\begin{matrix}\mathscr{D}_a \\ A \wedge B\end{matrix}}{A} \qquad\qquad \frac{\begin{matrix}\mathscr{D}_a \\ A \wedge B\end{matrix}}{B}$$

(where the left rule corresponds to case b, and the right rule corresponds to case c).

(ii) Clause (H3) justifies the implication introduction rule, when read from right to left: Suppose we have shown B directly or by (possibly repeatedly) using the assumption A. Then this means that we have found a construction that transforms a (hypothetical) proof of A into a proof of B. By clause (H3) this is a proof of the implication $A \to B$, which no longer depends on assumptions A:

$$\frac{\begin{matrix}[A] \\ \mathscr{D}_{a(b)} \\ B\end{matrix}}{A \to B}$$

When read from left to right, (H3) justifies the rule of implication elimination (i.e. *modus ponens*): Suppose we have shown $A \to B$. Then this means that we have found a construction that transforms proofs of A into a proof of B. If in addition we have shown A, then we obtain by an application of this construction to A the statement B:

$$\frac{\begin{matrix}\mathscr{D}_a & \mathscr{D}_b \\ A \to B & A\end{matrix}}{B}$$

(iii) We have already seen that by clause (H4) the principle *ex falso* $\bot \to A$ is valid under the BHK-interpretation. The corresponding rule is

$$\frac{\bot}{A}\,(\bot)$$

Note that the rule $(\bot)$ does not allow for the discharge of assumptions $\neg A$, in contradistinction to the classical rule of *reductio ad absurdum*

$$\frac{\begin{matrix}[\neg A] \\ \bot\end{matrix}}{A}\,(\bot)_c$$

The rule $(\bot)$ is therefore weaker than the rule $(\bot)_c$.

However, the classical rule $(\bot)_c$ cannot be justified by the BHK-interpretation, since it allows to show $A \vee \neg A$, which is not valid under the BHK-interpretation.

Remarks. (i) In the following we use *proposition letters* (also called *proposition variables*) $p, q, r, \ldots$ *proposition letters*

We refer to the *set of proposition letters* as $\mathrm{PV} := \{p, q, r, \ldots\}$. PV

(ii) As before, letters $A, B, C, \ldots$ are used as meta-variables for *formulas*, which are constructed from proposition letters with the logical constants $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\bot$ (*falsum*, absurdity). *formulas*

Proposition letters and $\bot$ are *atomic formulas* (short: *atoms*). *atomic formulas*

(iii) As usual, we define negation $\neg$ by implication and *falsum* as follows: $\neg A := A \rightarrow \bot$.

Note that this corresponds well with the BHK-interpretation of negation, where a proof of $\neg A$ consists in a construction which transforms any assumed proof of A into a proof of the absurdity $\bot$, for which there can be no proof by definition.

(iv) Moreover, we use letters $\Gamma, \Delta, \ldots$ to refer to sets of formulas.

Definition 1.1 (i) The *calculus NI of natural deduction (for intuitionistic logic)* is given by the following rules: *calculus NI*

Introduction rule	*Elimination rule*
$\dfrac{A_1 \quad A_2}{A_1 \wedge A_2}\,(\wedge\,\mathrm{I})$	$\dfrac{A_1 \wedge A_2}{A_i}\,(\wedge\,\mathrm{E})\ (i = 1 \text{ or } 2)$
$\dfrac{A_i}{A_1 \vee A_2}\,(\vee\,\mathrm{I})\ (i = 1 \text{ or } 2)$	$\dfrac{A_1 \vee A_2 \quad \begin{matrix}[A_1]\\ C\end{matrix} \quad \begin{matrix}[A_2]\\ C\end{matrix}}{C}\,(\vee\,\mathrm{E})$
$\dfrac{\begin{matrix}[A]\\ B\end{matrix}}{A \rightarrow B}\,(\rightarrow\mathrm{I})$	$\dfrac{A \rightarrow B \quad A}{B}\,(\rightarrow\mathrm{E})$
	Ex-falso rule
	$\dfrac{\bot}{A}\,(\bot)$

(ii) *Derivations in NI* are defined as usual.

(iii) If A is *derivable in NI* from a set of assumptions Γ, then we write $\Gamma \vdash_{\mathrm{NI}} A$. *derivable in NI*

(iv) If A is *provable in NI*, then we write $\vdash_{\mathrm{NI}} A$. *provable in NI*

Remarks. (i) The principle of *ex falso quodlibet sequitur* $\bot \rightarrow A$ could be rejected as being non-constructive by arguing that $\bot$ is a statement for which we just do not know yet whether it is provable. In this case we would not exclude the possibility that there could be a proof of $\bot$, e.g. if the currently accepted mathematics turns out to be inconsistent. Such an understanding leads to what is called *minimal logic*. *minimal logic*

We obtain the *calculus NM* for minimal logic by removing the ex-falso rule from NI. *calculus NM*

(ii) By replacing the ex-falso rule with the classical rule of *reductio ad absurdum* $(\bot)_c$ we obtain the *calculus NK* for classical logic. *calculus NK*

(iii) Intuitionistic and minimal logic are examples of non-classical (philosophical) logics that are weaker than classical logic; less logical laws hold in them than in classical logic. It is $\{A \mid \vdash_{\mathrm{NM}} A\} \subset \{A \mid \vdash_{\mathrm{NI}} A\} \subset \{A \mid \vdash_{\mathrm{NK}} A\}$.

Example. We show $\vdash_{\mathrm{NI}} \neg\neg(A \vee \neg A)$:

$$\dfrac{\dfrac{[\neg(A \vee \neg A)]^2 \qquad \dfrac{\dfrac{\dfrac{[\neg(A \vee \neg A)]^2 \qquad \dfrac{[A]^1}{A \vee \neg A}\,(\vee\,\mathrm{I})}{\bot}\,(\to\mathrm{E})}{\neg A}\,(\to\mathrm{I})^1}{A \vee \neg A}\,(\vee\,\mathrm{I})}{\bot}\,(\to\mathrm{E})}{\neg\neg(A \vee \neg A)}\,(\to\mathrm{I})^2$$

This illustrates that from a constructive point of view only *weak* counterexamples for *tertium non datur* $A \vee \neg A$ can be given. A *strong* counterexample would consist in showing that $p \vee \neg p$, for a certain statement p, leads to a contradiction; in other words, that $\neg(p \vee \neg p)$ holds. But this is not possible, since $\neg\neg(A \vee \neg A)$ holds for all statements A.

Remarks. (i) The formulas

$$A \to (B \to A) \qquad\qquad (\textit{ex quodlibet verum sequitur})$$

and

$$\neg A \to (A \to B) \qquad\qquad (\textit{ex falso quodlibet sequitur})$$

(resp. *ex contradictione quodlibet sequitur*), which are provable in NI, are sometimes called *paradoxes of implication*. If we consider their derivations

$$\dfrac{\dfrac{[A]^1}{B \to A}\,(\to\mathrm{I})}{A \to (B \to A)}\,(\to\mathrm{I})^1 \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\neg A]^2 \qquad [A]^1}{\bot}\,(\to\mathrm{E})}{B}\,(\bot)}{A \to B}\,(\to\mathrm{I})^1}{\neg A \to (A \to B)}\,(\to\mathrm{I})^2$$

we see that in both derivations B can be chosen arbitrarily. In the first derivation the formula B in $B \to A$ is in this sense not relevant for A, and in the second derivation the formula A in $A \to B$ is not relevant for B.

A logic in which neither $A \to (B \to A)$ nor $\neg A \to (A \to B)$ holds is called *relevance logic* (or *relevant logic*). *relevance logic*

(ii) Besides logical rules, i.e. rules which introduce or eliminate a logical constant, structural operations are of importance.

In the derivation of $A \to (B \to A)$ we went from the premiss A to the conclusion $B \to A$ without discharging an assumption B. This corresponds to the structural operation of *weakening*. *weakening*

In the derivation of $\neg\neg(A \vee \neg A)$ we discharged two occurrences of the assumption $\neg(A \vee \neg A)$ in one rule application. This corresponds to the structural operation of *contraction*. *contraction*

Another example is the proof of the *law of non-contradiction* $\neg(A \wedge \neg A)$, in which contraction is essential: *law of non-contradiction*

$$\dfrac{\dfrac{\dfrac{[A \wedge \neg A]^1}{\neg A}\,(\wedge\,\mathrm{E}) \qquad \dfrac{[A \wedge \neg A]^1}{A}\,(\wedge\,\mathrm{E})}{\bot}\,(\to\mathrm{E})}{\neg(A \wedge \neg A)}\,(\to\mathrm{I})^1$$

By imposing certain restrictions concerning structural operations one obtains *substructural logics* (cf. Došen & Schroeder-Heister, 1993). *substructural logics*

Theorem 1.2 *The derivability relation is transitive, i.e. the following holds: If* $\Gamma \vdash_{\mathrm{NI}} A$ *and* $\Delta, A \vdash_{\mathrm{NI}} B$*, then* $\Gamma, \Delta \vdash_{\mathrm{NI}} B$*.*

Proof. Assume $\Gamma \vdash_{\mathrm{NI}} A$ and $\Delta, A \vdash_{\mathrm{NI}} B$. Then there are derivations

$$\begin{array}{c} \Gamma \\ \mathscr{D} \\ A \end{array} \quad \text{and} \quad \begin{array}{c} \Delta, A \\ \mathscr{D}' \\ B \end{array}$$

Now we replace all assumptions A in the second derivation by the first derivation. We obtain the derivation

$$\begin{array}{c} \quad\;\Gamma \\ \quad\;\mathscr{D} \\ \Delta, A \\ \mathscr{D}' \\ B \end{array}$$

showing $\Gamma, \Delta \vdash_{\mathrm{NI}} B$. QED

Remark. In sequent calculus LI (see Gentzen, 1935) transitivity is made explicit by the rule

$$\frac{\Gamma \vdash A \qquad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,(\mathrm{Cut})$$

where A is called the *cut formula*.

1.4 Derivability and admissibility of rules

Besides the derivability of formulas the two notions of derivability and admissibility of rules are important. In contradistinction to classical logic, these two notions do not have the same extension in intuitionistic logic.

Definition 1.3 A rule *derivable rule*

$$\frac{A_1 \quad \dots \quad A_n}{B}$$

is called *derivable* in a calculus C, if $A_1, \dots, A_n \vdash_{\mathrm{C}} B$.

Example. The rule

$$\frac{A \vee \neg A}{\neg\neg A \to A}$$

is derivable in NI. (Exercise)

Definition 1.4 A rule R is called *admissible* in a calculus C, if the following holds: *admissible rule*

$$\text{If } \vdash_{\mathrm{C+R}} A, \text{ then } \vdash_{\mathrm{C}} A.$$

(Here $\vdash_{\mathrm{C+R}} A$ means that A is provable in the calculus C extended by the rule R, and $\vdash_{\mathrm{C}} A$ means that A is provable in C without R.)

Remarks. Let R be an arbitrary rule of the form $\dfrac{A_1 \quad \dots \quad A_n}{B}$.

(i) To demonstrate that R is derivable in NI we have to show

$$A_1, \ldots, A_n \vdash_{\mathrm{NI}} B$$

That is, we have to present a corresponding derivation.

(ii) To demonstrate that R is admissible in NI we have to show:

$$\text{If } \vdash_{\mathrm{NI}} A_1, \ldots, \vdash_{\mathrm{NI}} A_n, \text{ then } \vdash_{\mathrm{NI}} B.$$

(iii) The transitivity of the derivability relation $\vdash_{\mathrm{NI}}$ justifies the application of rules which are derivable in NI.

(iv) Derivability of a rule implies its admissibility.

(v) In the sequent calculus LI the (Cut) rule is admissible.

Example. To illustrate the difference between the derivability and the admissibility of rules we consider the following calculus for the generation of natural numbers $\mathbb{N}$:

$$\frac{}{0 \in \mathbb{N}}\,(1) \qquad\qquad \frac{k \in \mathbb{N}}{k' \in \mathbb{N}}\,(2)$$

The first rule is an axiom and says that 0 is a natural number. The second rule says that if k is a natural number, then its successor k' is a natural number as well.

(i) The rule

$$\frac{k \in \mathbb{N}}{k'' \in \mathbb{N}}\,(3)$$

is a derivable rule in this calculus, as witnessed by the derivation

$$\cfrac{\cfrac{k \in \mathbb{N}}{k' \in \mathbb{N}}\,(2)}{k'' \in \mathbb{N}}\,(2)$$

(ii) The rule

$$\frac{k' \in \mathbb{N}}{k \in \mathbb{N}}\,(4)$$

is an example of an admissible rule. Assuming that the premiss $k' \in \mathbb{N}$ is derivable, we have to show that the conclusion $k \in \mathbb{N}$ is also derivable.

The premiss cannot have been derived by rule (1), since k' cannot be 0. Hence the premiss must have been derived by rule (2). An application of (2) requires that there is a derivation of $k \in \mathbb{N}$. This is the desired derivation of the conclusion of (4).

However, rule (4) is obviously not derivable.

Remark. The notion of admissible rule is central in P. Lorenzen's (1915–1994) approach to logic (see Lorenzen, 1955), where we can also find the idea that the admissibility of a rule R in a calculus C has to be shown by an *elimination procedure* that eliminates every application of R from every derivation in the calculus C + R (cp. Gentzen's (1935) result on cut elimination for LI). We illustrate this idea in the proof of the following lemma.

Lemma 1.5 *Let $(\bot)^a$ be the restriction to atomic conclusions of the ex-falso rule $(\bot)$, and let $NI^a = NM + (\bot)^a$ (i.e. NI^a is the calculus obtained by replacing $(\bot)$ by $(\bot)^a$ in NI). The rule $(\bot)$ is admissible in NI^a.*

Proof. We have to show: If $\vdash_{\mathrm{NI}} A$, then $\vdash_{\mathrm{NI}^a} A$. Suppose $\vdash_{\mathrm{NI}} A$ is shown by the derivation

$$\begin{array}{c} \mathscr{D} \\ \dfrac{\bot}{C}\,(\bot) \\ \mathscr{D}' \\ A \end{array}$$

in which the exposed application of the ex-falso rule $(\bot)$ has conclusion C of arbitrary complexity, and where any other possibly occurring applications of $(\bot)$ have conclusions of lower complexity. We consider the structure of the formula C. (If the derivation of A in NI does not contain an application of $(\bot)$, then $\vdash_{\mathrm{NI}^a} A$ holds trivially.)

Induction base: C is atomic. Then the derivation has the form

$$\begin{array}{c} \mathscr{D} \\ \dfrac{\bot}{C}\,(\bot)^a \\ \mathscr{D}' \\ A \end{array}$$

(where the shown application of $(\bot)^a$ can be omitted, if C is $\bot$).

Induction hypothesis: The rule $(\bot)$ is admissible in NI^a for conclusions D and E.

Induction step: We have to show that $(\bot)$ is then also admissible for conclusions C of the form $\neg D$, $D \wedge E$, $D \vee E$ and $D \to E$.

We consider the case $D \wedge E$, i.e. the derivation

$$\begin{array}{c} \mathscr{D} \\ \dfrac{\bot}{D \wedge E}\,(\bot) \\ \mathscr{D}' \\ A \end{array}$$

Using the induction hypothesis we can transform this derivation into the derivation

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\mathscr{D}\\ \bot\end{array}}{D}\,(\bot) \qquad \dfrac{\begin{array}{c}\mathscr{D}\\ \bot\end{array}}{E}\,(\bot)}{D \wedge E}\,(\wedge\mathrm{I}) \\ \mathscr{D}' \\ A \end{array}$$

Hence the rule $(\bot)$ is also admissible for conclusions of the form $D \wedge E$.

Remaining cases as exercise. QED

Remark. The proof by induction shows how the complexity of the conclusion C of $(\bot)$ can be reduced step by step until C is finally atomic. Instead of $(\bot)$ we can thus use $(\bot)^a$ without reducing the strength of NI.

For NI one can show that there are admissible rules which are not derivable. Here we can make use of the fact that every derivation can be transformed into a normal form with certain useful properties.

1.5 Normalisability and properties of NI

Definition 1.6 (i) A formula occurrence in a derivation is called *maximal*, if it is the conclusion of an introduction rule and at the same time the major premiss of an elimination rule. The corresponding formula is called *maximal formula*. *maximal* *maximal formula*

(ii) Maximal formula occurrences can be eliminated by *reductions* which transform a derivation with a maximal formula occurrence into a derivation without that occurrence. *reductions*

For implicational maximal formulas $A \to B$ the $\to$-*reduction* ($\rhd_{\to}$) is defined as follows: *$\to$-reduction*

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \mathscr{D}\\ B\end{matrix}}{A \to B}(\to \mathrm{I}) \qquad \begin{matrix}\mathscr{D}'\\ A\end{matrix}}{B}(\to \mathrm{E}) \qquad \rhd_{\to} \qquad \begin{matrix}\mathscr{D}'\\ A\\ \mathscr{D}\\ B\end{matrix}$$

If $\mathscr{D}$ does not contain assumptions A, then the result of the $\to$-reduction is the derivation $\begin{matrix}\mathscr{D}\\ B\end{matrix}$.

Similarly, one can define $\wedge$- and $\vee$-reductions for maximal formulas of the form $A \wedge B$ and $A \vee B$. (Exercise)

(iii) A derivation is called *normal*, if it does not contain a maximal formula occurrence. In this case the derivation is in *normal form*. *normal* *normal form*

Remark. Besides reductions one has to consider certain permutations in addition; this is because of the form of the disjunction elimination rule, where the minor premisses and the conclusion are the same formula. However, we do not have to go into that here.

Theorem 1.7 (Normalisability)
If $\Gamma \vdash_{\mathrm{NI}} A$ holds, then there exists a normal derivation in NI of A from Γ.

Proof. See Prawitz (1965, Ch. IV § 1). QED

Corollary 1.8 *The following properties hold:*

(i) Subformula property: In a normal derivation of A from assumptions Γ every occurring formula is a subformula of A or a subformula of formulas in Γ. *subformula property*

(ii) Separation property: In a normal derivation of A from assumptions Γ there occur only rules dealing with the logical constants occurring in A and Γ. *separation property*

(iii) Derivations in normal form which do not contain open assumptions always end with an introduction rule.

(iv) Since $\bot$ cannot be derived by an introduction rule we have $\nvdash_{\mathrm{NI}} \bot$, i.e. NI is consistent. (One can argue likewise for every proposition letter.)

(v) Disjunction property: If $\vdash_{\mathrm{NI}} A \vee B$, then $\vdash_{\mathrm{NI}} A$ or $\vdash_{\mathrm{NI}} B$. *disjunction property*

(vi) Generalised disjunction property: Let Γ be a $\vee$-free set of formulas. Then the following holds: If $\Gamma \vdash_{\mathrm{NI}} A \vee B$, then $\Gamma \vdash_{\mathrm{NI}} A$ or $\Gamma \vdash_{\mathrm{NI}} B$. *generalised disjunction property*

Remarks. (i) Normalisability holds as well for NM and NK.

(ii) Every property mentioned in the corollary holds for NM, too. Note that in minimal logic $\bot$ is not a logical constant but is treated like a proposition letter.

(iii) Except for consistency (iv) none of the properties mentioned in the corollary holds for NK.

At least the subformula property holds in a restricted form; excluded are assumptions discharged in applications of *reductio ad absurdum*

$$\frac{\begin{array}{c}[\neg A]\\ \bot\end{array}}{A}\,(\bot)_c$$

and occurrences of $\bot$ immediately below such assumptions.

A counterexample for the separation property is given by Peirce's law

$$((A \to B) \to A) \to A$$

Any proof of this purely implicative formula requires besides $(\to \mathrm{I})$ and $(\to \mathrm{E})$ also *reductio ad absurdum* $(\bot)_c$.

(iv) For the generalised disjunction property (vi) it is sufficient to make the weaker presupposition that no formula in Γ contains a strictly positive subformula with main connective $\vee$. (See Prawitz, 1965, p. 43, 55.)

Definition 1.9 Bi-implication (or equivalence) $\leftrightarrow$ is defined as usual:

$$A \leftrightarrow B := (A \to B) \wedge (B \to A)$$

Bi-implication $\leftrightarrow$ shall bind as strongly as $\to$.

Theorem 1.10 *There are admissible but non-derivable rules in NI.*

Proof. *Harrop's rule* *Harrop's rule*

$$\frac{\neg A \to (B \vee C)}{(\neg A \to B) \vee (\neg A \to C)}$$

is admissible in NI, but not derivable.
Admissibility follows from the generalised disjunction property and the fact that a negated formula $\neg A$ can always be transformed into a $\vee$-free formula A' and vice versa, such that $\vdash_{\mathrm{NI}} \neg A \leftrightarrow A'$ holds.
Suppose $\vdash_{\mathrm{NI}} \neg A \to (B \vee C)$ holds due to a derivation $\mathscr{D}$ in normal form. By Corollary 1.8 (iii) this derivation $\mathscr{D}$ ends with $(\to \mathrm{I})$, i.e. $\neg A \vdash_{\mathrm{NI}} B \vee C$ holds, too. Now we transform $\neg A$ using

$$(*)\left\{\begin{array}{l}\vdash_{\mathrm{NI}} \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)\\ \vdash_{\mathrm{NI}} \neg(A \wedge B) \leftrightarrow \neg(\neg\neg A \wedge \neg\neg B)\\ \vdash_{\mathrm{NI}} \neg(A \to B) \leftrightarrow (\neg\neg A \wedge \neg B)\end{array}\right.$$

into a $\vee$-free formula A', such that $\vdash_{\mathrm{NI}} \neg A \leftrightarrow A'$. We thus obtain $A' \vdash_{\mathrm{NI}} B \vee C$, to which we can apply the generalised disjunction property in order to obtain $A' \vdash_{\mathrm{NI}} B$ or $A' \vdash_{\mathrm{NI}} C$. Using $(*)$ again yields $\neg A \vdash_{\mathrm{NI}} B$ and $\neg A \vdash_{\mathrm{NI}} C$, respectively. In both cases there must be corresponding derivations $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively, which we can extend to a proof of the conclusion of Harrop's rule:

$$\dfrac{\dfrac{\begin{matrix}[\neg A]^1\\ \mathscr{D}_1\\ B\end{matrix}}{\neg A \to B}\,(\to \mathrm{I})^1}{(\neg A \to B) \vee (\neg A \to C)}\,(\vee\,\mathrm{I}) \qquad \text{respectively} \qquad \dfrac{\dfrac{\begin{matrix}[\neg A]^1\\ \mathscr{D}_2\\ C\end{matrix}}{\neg A \to C}\,(\to \mathrm{I})^1}{(\neg A \to B) \vee (\neg A \to C)}\,(\vee\,\mathrm{I})$$

We have thus shown: If $\vdash_{\mathrm{NI}} \neg A \to (B \vee C)$, then $\vdash_{\mathrm{NI}} (\neg A \to B) \vee (\neg A \to C)$. That is, Harrop's rule is admissible in NI.
We do not prove non-derivability of Harrop's rule here. A proof can be given by a counterexample in Kripke semantics (see Section 1.7), for which NI is sound (and complete). QED

Remarks. (i) An admissible but non-derivable rule may only be used in a derivation if none of its premisses depends on open assumptions. Otherwise one could derive formulas in NI which are not provable:

$$\dfrac{\dfrac{[\neg A \to (B \vee C)]^1}{(\neg A \to B) \vee (\neg A \to C)}\,(\text{Harrop's rule})\ \lightning}{(\neg A \to (B \vee C)) \to ((\neg A \to B) \vee (\neg A \to C))}\,(\to \mathrm{I})^1$$

In this (incorrect) application of Harrop's rule its premiss still depends on itself as an open assumption; one would obtain the *Kreisel–Putnam formula* $(\neg A \to (B \vee C)) \to ((\neg A \to B) \vee (\neg A \to C))$, which is, however, not provable in NI. *Kreisel–Putnam formula*

(ii) If one extends NI by adding admissible but non-derivable rules, then one obtains so-called *superintuitionistic* or *intermediate logics*, which are located between intuitionistic and classical logic in strength. *intermediate logics*

For example, if one extends NI by adding Harrop's rule, then one obtains the so-called *Kreisel–Putnam logic*. Another example is the so-called *Gödel–Dummett logic*, which one obtains by adding $(A \to B) \vee (B \to A)$ as an axiom.

There are infinitely many intermediate logics. (See Gödel, 1932.)

Examples. Further examples of rules that are admissible but not derivable in NI are:

(i) The rule $\dfrac{\neg\neg A \to A}{A \vee \neg A}$

(ii) *Mints's rule*: $\dfrac{(A \to B) \to (A \vee C)}{((A \to B) \to A) \vee ((A \to B) \to C)}$ *Mints's rule*

Remarks. (i) Logics in which not every admissible rule is derivable are also called *structurally incomplete*. *structurally incomplete*

(ii) In contradistinction to NI every rule admissible in NK is also derivable, i.e. classical logic is *structurally complete*. *structurally complete*

It is easy to show this (classically) by contraposition: Suppose the rule

$$\dfrac{A_1 \quad \ldots \quad A_n}{B}$$

is not derivable in NK, i.e. $A_1, \ldots, A_n \nvdash_{\mathrm{NK}} B$. Then by completeness $A_1, \ldots, A_n \nvDash B$ (for the semantic consequence relation $\vDash$ of classical logic). Hence there must exist a valuation v, such that $[\![A_1]\!]^v = \ldots = [\![A_n]\!]^v = \mathsf{true}$, but $[\![B]\!]^v = \mathsf{false}$. Now we replace all proposition letters $A \in \mathrm{PV}$ in $A_1, \ldots, A_n, B$ either by $\top := p \to p$, if $v(A) = \mathsf{true}$, or by $\bot$, if $v(A) = \mathsf{false}$. Then $\vDash A_1, \ldots, \vDash A_n$ and $\vDash \neg B$, and hence especially $\nvDash B$. By completeness and soundness of NK therefore $\vdash_{\mathrm{NK}} A_1, \ldots, \vdash_{\mathrm{NK}} A_n$, but $\nvdash_{\mathrm{NK}} B$, i.e. the rule cannot be admissible.

Definition 1.11 We say that a logical constant $* \in \{\neg, \wedge, \vee, \to\}$ can be *expressed* by a formula F in a calculus C, if $\vdash_{\mathrm{C}} *A \leftrightarrow F$ or $\vdash_{\mathrm{C}} (A * B) \leftrightarrow F$ for a formula F, in which $*$ itself does not occur. If no such formula F exists, then $*$ is called *independent*. *independent*

Theorem 1.12 *In NI each of the logical constants $\neg$, $\wedge$, $\vee$ and $\to$ is independent.*

Proof. See Wajsberg (1938) or McKinsey (1939). QED

Remarks. (i) No proper subset of $\{\neg, \wedge, \vee, \to\}$ can be functionally complete for intuitionistic logic. This is another essential difference w.r.t. classical logic, in which e.g. $\{\neg, \wedge\}$, $\{\neg, \vee\}$ and $\{\neg, \to\}$ are functionally complete sets.

In this sense each of the logical constants $\neg$, $\wedge$, $\vee$ and $\to$ has its distinct meaning in intuitionistic logic, since none can be expressed by the respective others.

(ii) A ternary Sheffer function t for the set $\{\neg, \wedge, \vee, \to\}$ is presented in Došen (1985):

$$t(A, B, C) := (A \vee B) \leftrightarrow (C \leftrightarrow \neg B)$$

(iii) In Gödel–Dummett logic GD (also referred to as G or LC in the literature) at least $\vee$ can be expressed using $\{\wedge, \to\}$. We have (exercise):

$$\vdash_{\mathrm{GD}} (A \vee B) \leftrightarrow (((A \to B) \to B) \wedge ((B \to A) \to A))$$

This intermediate logic thus also lies w.r.t. the independence of logical constants between intuitionistic and classical logic.

Theorem 1.13 (Deduction theorem)
$A_1, \ldots, A_n \vdash_{\mathrm{NI}} B \iff \vdash_{\mathrm{NI}} A_1 \wedge \ldots \wedge A_n \to B$. *(Likewise for NM and NK.)*

Proof. Exercise. QED

1.6 On the relation between classical and minimal/intuitionistic logic

In the following we investigate the relation between classical logic (NK) and minimal logic (NM) as well as intuitionistic logic (NI) in more detail. As a technical means one uses certain translations of formulas. As a further means we use a result on so-called negative formulas, which constitute a fragment of our language of propositional logic.

Definition 1.14 A formula A is called *negative*, if it is $\vee$-free, and if every occurrence of proposition letters is negated. *negative formula*

Lemma 1.15 *For negative formulas A we have: $\vdash_{\mathrm{NM}} \neg\neg A \leftrightarrow A$.*

Proof. We show this by an induction on the structure of A, where we make use of the fact that for arbitrary formulas A, B the following holds:

$$\vdash_{\mathrm{NM}} A \to \neg\neg A \tag{1}$$
$$\vdash_{\mathrm{NM}} \neg\neg\neg A \leftrightarrow \neg A \tag{2}$$
$$\vdash_{\mathrm{NM}} \neg\neg(A \wedge B) \to (\neg\neg A \wedge \neg\neg B) \tag{3}$$
$$\vdash_{\mathrm{NM}} \neg\neg(A \to B) \to (A \to \neg\neg B) \tag{4}$$

(In NI the two formulas $(\neg\neg A \wedge \neg\neg B) \to \neg\neg(A \wedge B)$ and $(A \to \neg\neg B) \to \neg\neg(A \to B)$ hold as well; however, we do not need this here.)

Induction base: If A is a proposition letter, then A cannot be a negative formula; the assertion thus follows trivially.

In case $A \equiv \bot$ (i.e., A is syntactically identical to $\bot$) the assertion follows from

$$\dfrac{(\bot \to \bot) \to \bot \qquad \dfrac{[\bot]^1}{\bot \to \bot}\,(\to \mathrm{I})^1}{\bot}\,(\to \mathrm{E})$$

and (1).

Induction hypothesis: The assertion holds for B and C.

Induction step: We have to show that the assertion then holds for $\neg B$, $B \wedge C$, $B \vee C$ and $B \to C$, too.

In case $A \equiv B \vee C$ the assertion follows trivially, since A is not a negative formula.

Now we consider the case $A \equiv (B \to C)$: It is $\neg\neg A \equiv \neg\neg(B \to C)$. By (4) we have

$$\vdash_{\mathrm{NM}} \neg\neg(B \to C) \to (B \to \neg\neg C)$$

and the deduction theorem yields

$$\neg\neg(B \to C) \vdash_{\mathrm{NM}} B \to \neg\neg C \tag{A}$$

By induction hypothesis we have as a special case $\vdash_{\mathrm{NM}} \neg\neg C \to C$, i.e. the rule

$$\dfrac{\neg\neg C}{C}\,(*)$$

is derivable in NM as long as C is a negative formula. With

$$\dfrac{\dfrac{\dfrac{B \to \neg\neg C \qquad [B]^1}{\neg\neg C}\,(\to \mathrm{E})}{C}\,(*)}{B \to C}\,(\to \mathrm{I})^1$$

one obtains

$$B \to \neg\neg C \vdash_{\mathrm{NM}} B \to C \tag{B}$$

Transitivity of $\vdash_{\mathrm{NM}}$ applied to (A) and (B) yields $\neg\neg(B \to C) \vdash_{\mathrm{NM}} B \to C$, from which we get $\vdash_{\mathrm{NM}} \neg\neg(B \to C) \to (B \to C)$ by using the deduction theorem. The assertion follows with (1).

Remaining cases as exercise. QED

Next we consider a translation of formulas in which proposition letters are negated twice, and in which disjunctive formulas obtain a weaker meaning by expressing them using negation and conjunction.

Remark. It is $A \vee B \vdash_{\mathrm{NM}} \neg(\neg A \wedge \neg B)$, but $\neg(\neg A \wedge \neg B) \nvdash_{\mathrm{NI}} A \vee B$.

translation $^{\mathrm{g}}$

Definition 1.16 The *translation* $^{\mathrm{g}}$ is defined as follows:

(i) $\perp^{\mathrm{g}} := \perp$,

(ii) $A^{\mathrm{g}} := \neg\neg A$, if A is a proposition letter,

(iii) $(A \wedge B)^{\mathrm{g}} := A^{\mathrm{g}} \wedge B^{\mathrm{g}}$,

(iv) $(A \vee B)^{\mathrm{g}} := \neg(\neg A^{\mathrm{g}} \wedge \neg B^{\mathrm{g}})$,

(v) $(A \to B)^{\mathrm{g}} := A^{\mathrm{g}} \to B^{\mathrm{g}}$.

For sets of formulas Γ let $\Gamma^{\mathrm{g}} := \{B^{\mathrm{g}} \mid B \in \Gamma\}$.

negative translations

Remark. The translation $^{\mathrm{g}}$ goes in this form back to G. Gentzen (1909–1945). Alternative translations were used by Kolmogorov, Gödel, Kuroda and Krivine. In general such translations are called *negative translations*.

Examples. (i) $(p \vee \neg p)^{\mathrm{g}} \equiv \neg(\neg p^{\mathrm{g}} \wedge \neg\neg p^{\mathrm{g}}) \equiv \neg(\neg\neg\neg p \wedge \neg\neg\neg\neg p)$.

(ii) $(\neg\neg p \to p)^{\mathrm{g}} \equiv \neg\neg p^{\mathrm{g}} \to p^{\mathrm{g}} \equiv \neg\neg\neg\neg p \to \neg\neg p$.

By using $\vdash_{\mathrm{NM}} \neg\neg\neg A \leftrightarrow \neg A$ (sub)formulas $\neg\neg\neg A$ can be further simplified into $\neg A$.

Theorem 1.17 $\Gamma \vdash_{\mathrm{NK}} A \iff \Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} A^{\mathrm{g}}$.

Proof. To prove the direction from right to left one first shows $\vdash_{\mathrm{NK}} A \leftrightarrow A^{\mathrm{g}}$, and then uses: $\Gamma \vdash_{\mathrm{NM}} A \implies \Gamma \vdash_{\mathrm{NK}} A$. (Exercise)

We prove the direction from left to right by an induction on the structure of derivations $\mathscr{D}$ of the formula A from the set of assumptions Γ.

Induction base: Let $A \in \Gamma$; then also $A^{\mathrm{g}} \in \Gamma^{\mathrm{g}}$. Hence $\Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} A^{\mathrm{g}}$. (This includes the case $\{A\} = \Gamma$, where the derivation $\mathscr{D}$ is just the node A.)

Induction hypothesis: The assertion holds for the derivation(s) of the premiss(es) of the last rule application in $\mathscr{D}$.

Induction step: We have to consider all rules which are applicable in the last step. As examples we treat the cases $(\to\mathrm{I})$, $(\vee\,\mathrm{E})$ and $(\perp)_c$.

(i) $\mathscr{D}$ ends with $(\to\mathrm{I})$:

$$\dfrac{\begin{array}{c}\Gamma, [A]^n \\ \mathscr{D} \\ B\end{array}}{A \to B}\,(\to\mathrm{I})^n$$

By the induction hypothesis it holds: $\Gamma^{\mathrm{g}}, A^{\mathrm{g}} \vdash_{\mathrm{NM}} B^{\mathrm{g}}$. With

$$\dfrac{\begin{array}{c}\Gamma^{\mathrm{g}}, [A^{\mathrm{g}}]^n \\ \mathscr{D}^{\mathrm{g}} \\ B^{\mathrm{g}}\end{array}}{A^{\mathrm{g}} \to B^{\mathrm{g}}}\,(\to\mathrm{I})^n$$

we have $\Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} A^{\mathrm{g}} \to B^{\mathrm{g}}$, and by definition of $^{\mathrm{g}}$ also $\Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} (A \to B)^{\mathrm{g}}$.

(ii) $\mathscr{D}$ ends with $(\vee\,\mathrm{E})$:

$$\dfrac{\begin{array}{c}\Gamma\\ \mathscr{D}\\ A\vee B\end{array} \qquad \begin{array}{c}\Gamma,[A]^n\\ \mathscr{D}_1\\ C\end{array} \qquad \begin{array}{c}\Gamma,[B]^n\\ \mathscr{D}_2\\ C\end{array}}{C}\,(\vee\,\mathrm{E})^n$$

By the induction hypothesis:

(1) $\Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} (A\vee B)^{\mathrm{g}}$, (2) $\Gamma^{\mathrm{g}}, A^{\mathrm{g}} \vdash_{\mathrm{NM}} C^{\mathrm{g}}$, (3) $\Gamma^{\mathrm{g}}, B^{\mathrm{g}} \vdash_{\mathrm{NM}} C^{\mathrm{g}}$.

By definition of ${}^{\mathrm{g}}$ and with (1) it then holds: $\Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} \neg(\neg A^{\mathrm{g}} \wedge \neg B^{\mathrm{g}})$, by a derivation $\mathscr{D}'$. Moreover, due to (2) and (3) there exist derivations

$$\mathscr{D}'_1 \left\{ \begin{array}{c}\Gamma^{\mathrm{g}},[A^{\mathrm{g}}]^n\\ \mathscr{D}_1^{\mathrm{g}}\\ \dfrac{C^{\mathrm{g}}}{A^{\mathrm{g}}\to C^{\mathrm{g}}}\,(\to\mathrm{I})^n\end{array}\right. \qquad \text{and} \qquad \mathscr{D}'_2 \left\{ \begin{array}{c}\Gamma^{\mathrm{g}},[B^{\mathrm{g}}]^n\\ \mathscr{D}_2^{\mathrm{g}}\\ \dfrac{C^{\mathrm{g}}}{B^{\mathrm{g}}\to C^{\mathrm{g}}}\,(\to\mathrm{I})^n\end{array}\right.$$

Then $\Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} C^{\mathrm{g}}$ holds by the following derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Gamma^{\mathrm{g}}\\ \mathscr{D}'\\ \neg(\neg A^{\mathrm{g}}\wedge\neg B^{\mathrm{g}})\end{array} \qquad \dfrac{\dfrac{\dfrac{[\neg C^{\mathrm{g}}]^3 \qquad \dfrac{\begin{array}{c}\mathscr{D}'_1\\ A^{\mathrm{g}}\to C^{\mathrm{g}}\end{array} \quad [A^{\mathrm{g}}]^1}{C^{\mathrm{g}}}\,(\to\mathrm{E})}{\bot}\,(\to\mathrm{E})}{\neg A^{\mathrm{g}}}\,(\to\mathrm{I})^1 \qquad \dfrac{\dfrac{[\neg C^{\mathrm{g}}]^3 \qquad \dfrac{\begin{array}{c}\mathscr{D}'_2\\ B^{\mathrm{g}}\to C^{\mathrm{g}}\end{array} \quad [B^{\mathrm{g}}]^2}{C^{\mathrm{g}}}\,(\to\mathrm{E})}{\bot}\,(\to\mathrm{E})}{\neg B^{\mathrm{g}}}\,(\to\mathrm{I})^2}{\neg A^{\mathrm{g}}\wedge\neg B^{\mathrm{g}}}\,(\wedge\,\mathrm{I})}{\bot}\,(\to\mathrm{E})}{\neg\neg C^{\mathrm{g}}}\,(\to\mathrm{I})^3}{C^{\mathrm{g}}}\,(\text{Lemma 1.15})$$

In the last step we were able to apply Lemma 1.15, since C^{g} is a negative formula: it is $\vee$-free (Def. 1.16 (iv)), and only (double) negated proposition letters do occur (Def. 1.16 (ii)).

(iii) $\mathscr{D}$ ends with $(\bot)_c$:

$$\begin{array}{c}\Gamma,[\neg A]^n\\ \mathscr{D}\\ \dfrac{\bot}{A}\,(\bot)_c^n\end{array}$$

By induction hypothesis: $\Gamma^{\mathrm{g}}, (\neg A)^{\mathrm{g}} \vdash_{\mathrm{NM}} \bot^{\mathrm{g}}$; and with

$$\bot^{\mathrm{g}} \equiv \bot \quad \text{and} \quad (\neg A)^{\mathrm{g}} \equiv (A\to\bot)^{\mathrm{g}} \equiv A^{\mathrm{g}}\to\bot^{\mathrm{g}} \equiv A^{\mathrm{g}}\to\bot \equiv \neg A^{\mathrm{g}}$$

also $\Gamma^{\mathrm{g}}, \neg A^{\mathrm{g}} \vdash_{\mathrm{NM}} \bot$ holds, by a derivation $\mathscr{D}^{\mathrm{g}}$. By Lemma 1.15 we have in addition $\vdash_{\mathrm{NM}} \neg\neg A^{\mathrm{g}} \to A^{\mathrm{g}}$, since A^{g} is a negative formula. Then $\Gamma^{\mathrm{g}} \vdash_{\mathrm{NM}} A^{\mathrm{g}}$ holds due to the derivation

$$\dfrac{\dfrac{}{\neg\neg A^{\mathrm{g}}\to A^{\mathrm{g}}}\,(\text{Lemma 1.15}) \qquad \begin{array}{c}\Gamma,[\neg A^{\mathrm{g}}]^n\\ \mathscr{D}^{\mathrm{g}}\\ \dfrac{\bot}{\neg\neg A^{\mathrm{g}}}\,(\to\mathrm{I})^n\end{array}}{A^{\mathrm{g}}}\,(\to\mathrm{E})$$

Remaining cases as exercise. QED

Corollary 1.18 *For negative formulas A we have:* $\vdash_{\text{NK}} A \iff \vdash_{\text{NM}} A$.

Proof. For negative formulas A also A^{g} is a negative formula, in which every proposition letter is prefixed by two additional negations. Therefore by Lemma 1.15 (and (1)-(4) in its proof) it holds: $\vdash_{\text{NM}} A \leftrightarrow A^{\text{g}}$. QED

Remarks. (i) Classical logic is thus a conservative extension of minimal logic w.r.t. negative formulas.

(ii) In classical logic every formula is equivalent to a negative formula: replace each proposition letter $A \in \text{PV}$ by $\neg\neg A$, and remove disjunctions with De Morgan. Hence classical logic is in a certain sense contained in minimal logic, although $\vdash_{\text{NK}} A \implies \vdash_{\text{NM}} A$ does not hold for arbitrary formulas A.

(iii) Since $\bot$ is a negative formula, Corollary 1.18 yields: $\nvdash_{\text{NK}} \bot \iff \nvdash_{\text{NM}} \bot$. Thus NK is consistent iff NM is consistent.

Consistency of NK follows directly from $\nvDash \bot$ and soundness. Hence NM is consistent as well. The latter already follows from the consistency of NI (Corollary 1.8 (iv)), since NM is contained in NI.

Theorem 1.19 (Glivenko) *It holds:*

(i) $\vdash_{\text{NK}} A \iff \vdash_{\text{NI}} \neg\neg A$.

(ii) $\vdash_{\text{NK}} \neg A \iff \vdash_{\text{NI}} \neg A$.

Proof. (i) One shows by an induction on the structure of A that $\vdash_{\text{NI}} A^{\text{g}} \leftrightarrow \neg\neg A$ (exercise). The assertion follows with Theorem 1.17.

(ii) Suppose $\vdash_{\text{NK}} \neg A$. Then it follows with (i) that $\vdash_{\text{NI}} \neg\neg\neg A$. With $\vdash_{\text{NI}} \neg\neg\neg A \leftrightarrow \neg A$ we can conclude $\vdash_{\text{NI}} \neg A$.

(The direction from right to left follows for both assertions already from the fact that NK is an extension of NI, and $\vdash_{\text{NK}} \neg\neg A \to A$.) QED

Remarks. (i) The following generalisation holds as well:

$$A_1, \ldots, A_n \vdash_{\text{NK}} A \iff \neg\neg A_1, \ldots, \neg\neg A_n \vdash_{\text{NI}} \neg\neg A$$

(ii) Glivenko's Theorem does *not* hold for first-order logic. For calculi NK and NI that are extended to first-order logic we have e.g. $\vdash_{\text{NK}} \forall x(A(x) \vee \neg A(x))$, but $\nvdash_{\text{NI}} \neg\neg\forall x(A(x) \vee \neg A(x))$.

1.7 Kripke-semantics for intuitionistic logic

Gödel (1932) showed that there cannot be a finite-valued truth-conditional semantics for intuitionistic logic.

In the following we consider Kripke-semantics for intuitionistic logic. Kripke-semantics is a so-called *possible-worlds semantics*, which is not truth-conditional. The calculus NI is sound and complete for this semantics.

possible-worlds semantics

In order to motivate this semantics we first consider the actions of an *idealised mathematician*, who is in intuitionism also referred to as a *creating subject*; we then observe the situation in the presence of a weak counterexample for $A \vee \neg A$.

At a given moment the idealised mathematician has a certain knowledge; that is, there is a set of assertions which are accepted as being valid at that moment. Over time, our mathematician can extend this knowledge in different directions. The possible states of the mathematician that correspond to different extensions of knowledge should thus not be understood as being linearly ordered; instead these states form a *partial order*. A partial order is reflexive (each state is related to itself), antisymmetric (two different states cannot at the same time occur before the respective other) and transitive; the order is partial, since two different states need not necessarily be related to each other (there exist alternative extensions of knowledge). The knowledge of the idealised mathematician is assumed to be *monotone* w.r.t. later moments in time. This property is also called *persistence*, i.e. knowledge cannot get lost.

partial order

monotony

The idea of a temporal order of states serves only as an illustration here. What is essential is the idea that states which represent knowledge suggest a partial order with monotony. The logical constants are now understood in such a way that the interpretation of complex statements depends on the interpretations of its sub-statements:

- For example, if the idealised mathematician has in state k accepted p as valid and has accepted also q as valid, then $p \wedge q$ is valid in k as well.
- Correspondingly, the statement $p \vee q$ is valid in k iff the statement p is valid in p or the statement q is valid in k.

An implication $p \rightarrow q$ might also be seen to be valid in a state k if in k it is neither known whether p is valid nor whether q is valid. For example, let p be the statement “a series of 1000 ones occurs in the decimal expansion of π”, and let q be the statement “a series of 999 ones occurs in the decimal expansion of π”. Even if we do neither know p nor q in state k, we nevertheless know that $p \rightarrow q$ must hold in k. Now we consider a state k' that extends our knowledge by p. Due to monotony $p \rightarrow q$ holds in k' as well; consequently, also q must hold in k'. On the other hand we have that an implication $p \rightarrow q$ holds in a state k, if in every extension of k, in which p holds, also q holds. By reflexivity this includes k as a trivial extension.

- Thus in a state k an implication $p \rightarrow q$ holds iff in every extension k' of k (including k) we have: If p holds, then q holds as well.
- There is no state k in which $\perp$ holds. This follows in the described setting from the fact that our idealised mathematician has knowledge at any given moment in time.

In order to illustrate this we now consider the situation for a weak counterexample for $A \vee \neg A$. We presuppose that in the present state k_0 the statement p is still undecided. It is, however, not impossible that a proof of p is found at a later moment (state k_1). This situation can be depicted as follows (where we write $\Vdash p$ for “p holds”, and where we omit arrows representing reflexivity):

$$\begin{tikzcd} \boxed{k_1} \Vdash p \\ \boxed{k_0} \arrow[u] \end{tikzcd}$$

In state k_0 we neither know whether p holds nor whether p does not hold. In k_0 we are also unable to assert $\neg p$, since there can be a state in which p holds (namely k_1). Thus

also $p \vee \neg p$ cannot hold in state k_0. However, $\neg\neg p$ holds in k_0, since there is no state after k_0 in which $\neg p$ would hold.
Since p is still undecided in k_0, it could happen, however, that at a later moment k_2 we find a proof of $\neg p$. We therefore have the following situation:

$$\begin{tikzcd} \boxed{k_1} \Vdash p & & \boxed{k_2} \Vdash \neg p \\ & \boxed{k_0} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

We still do not know in k_0 whether $\neg p$ holds. Moreover, $\neg\neg p$ cannot hold in k_0. In this case, $\perp$ would also have to hold in every state after k_2 (including k_2), which cannot be the case. Hence, also $\neg\neg p \vee \neg p$ does not hold in k_0.

We now give a formal definition of Kripke-semantics. We use the notion of a *model* in the neutral sense of *structure*; that is, a model describes a certain situation in which a formula A can either be valid or invalid. In the first case we will say that A is valid in the considered model, and in the second case that the model is a countermodel for A.

Definition 1.20 A *Kripke-model* is a triple $\mathscr{K} := \langle K, \leq, \Vdash \rangle$, consisting of a *frame* $\langle K, \leq \rangle$ and a *valuation* $\Vdash$. *Kripke-model*

- The *frame* $\langle K, \leq \rangle$ comprises a non-empty *set of states* K and a *partial order relation* $\leq$ on $K \times K$. *frame*

  (Hence the relation $\leq$ is reflexive, antisymmetric and transitive; it is partial, since not all elements of K need to relate to each other.)

  We call the elements of K *states* $k_0, k_1, \dots, k, k', k'', \dots$ *states*

  Frames $\langle K, \leq \rangle$ are thus non-empty partially ordered sets of states.

  (States are also called *nodes* or *possible worlds*. We also write $k' \geq k$ instead of $k \leq k'$; one says e.g. "k' extends k" or "k sees k'".)

- A *valuation* $\Vdash$ (read: *forces*; *forcing relation*) is a relation on $K \times \mathrm{PV}$, i.e. between states $k \in K$ and proposition letters $A \in \mathrm{PV}$, which obeys the following *monotony condition*: *valuation* *monotony condition*

  $$\text{If } k \Vdash A \text{ and } k' \geq k, \text{ then } k' \Vdash A.$$

For formulas that are not proposition letters we extend the forcing relation $\Vdash$ by the following clauses:

$$\begin{aligned}
k \Vdash A \wedge B \ &:\Longleftrightarrow\ k \Vdash A \text{ and } k \Vdash B \\
k \Vdash A \vee B \ &:\Longleftrightarrow\ k \Vdash A \text{ or } k \Vdash B \\
k \Vdash A \to B \ &:\Longleftrightarrow\ \text{For all } k' \geq k\text{: if } k' \Vdash A, \text{ then } k' \Vdash B \\
\text{not } k \Vdash \perp &
\end{aligned}$$

(The last clause is equivalent to $k \nVdash \perp$; that is, there is no element k in K, such that $k \Vdash \perp$.)

A formula A is called *valid in k* iff $k \Vdash A$. *valid in k*

Lemma 1.21 *(i)* $k \Vdash \neg A$ *iff for all states* $k' \geq k$*:* $k' \nVdash A$.

(ii) $k \Vdash \neg\neg A$ *iff there exists for all states* $k' \geq k$ *a state* $k'' \geq k'$*, such that* $k'' \Vdash A$.

Proof. (i) It is

$$\begin{aligned} k \Vdash \neg A &\iff k \Vdash A \to \bot \\ &\iff \text{for all } k' \geq k\text{: if } k' \Vdash A\text{, then } k' \Vdash \bot \qquad (*) \\ &\iff \text{for all } k' \geq k\text{: } k' \nVdash A \end{aligned}$$

In the last step, "$\Longrightarrow$" holds, since in every model for each k: $k \nVdash \bot$. (From $(*)$ follows

$$\text{for all } k' \geq k\text{: if } k' \nVdash \bot\text{, then } k' \nVdash A$$

from which we obtain with $k \nVdash \bot$ (for all k): $k' \nVdash A$, for all $k' \geq k$.) The opposite direction is obtained with *ex falso*.

(ii) It is

$$\begin{aligned} k \Vdash \neg\neg A &\iff \text{for all } k' \geq k\text{: } k' \nVdash \neg A \\ &\iff \text{for all } k' \geq k \text{ it does not hold that for all } k'' \geq k'\text{: } k'' \nVdash A \\ &\iff \text{for all } k' \geq k \text{ there exists a } k'' \geq k'\text{, such that } k'' \Vdash A \end{aligned}$$

In the last step, "$\Longrightarrow$" holds only classically (as long as we do not restrict ourselves to finite models). QED

Remark. The comment at the end of the proof of (ii) indicates that it can make a difference whether Kripke-semantics is treated from a classical or from a constructivistic point of view.

Lemma 1.22 (Monotony) *For all* $k, k' \in K$ *we have: If* $k \Vdash A$ *and* $k' \geq k$*, then* $k' \Vdash A$*. (That is, the monotony condition that we have imposed on proposition letters holds as well for arbitrary formulas* A*.)*

Proof. By induction on the structure of A.

Induction base: Let A be atomic. If $A \equiv \bot$, then the assertion holds trivially, since $k \nVdash \bot$. If A is a proposition letter, then the assertion holds due to the monotony condition in Definition 1.20.

Induction hypothesis: The assertion holds for formulas B and C.

Induction step: Case $A \equiv B \wedge C$: Assume $k \Vdash B \wedge C$ and $k' \geq k$. We have $k \Vdash B \wedge C$ iff $k \Vdash B$ and $k \Vdash C$. By the induction hypothesis we have then also $k' \Vdash B$ and $k' \Vdash C$, hence $k' \Vdash B \wedge C$.

Case $A \equiv B \vee C$: Analogously to the former case.

Case $A \equiv B \to C$: Assume $k \Vdash B \to C$ and $k' \geq k$. Consider an arbitrary state k'', such that $k'' \geq k'$ and $k'' \Vdash B$. By transitivity of $\leq$ we have $k'' \geq k$. Since $k \Vdash B \to C$ also $k'' \Vdash C$ must hold. Therefore for all states $k'' \geq k'$: If $k'' \Vdash B$, then $k'' \Vdash C$, i.e. $k' \Vdash B \to C$. QED

Definition 1.23 Let $\mathcal{K} := \langle K, \leq, \Vdash \rangle$ be a Kripke-model. We define

Validity in a model: *validity in a model*

$$\mathcal{K} \Vdash A \;:\Longleftrightarrow\; \text{For all } k \in K\text{: } k \Vdash A$$

Kripke-validity: *Kripke-validity*

$$\Vdash A \;:\Longleftrightarrow\; \text{For all models } \mathcal{K}\text{: } \mathcal{K} \Vdash A$$

Remark. If k_0 is the smallest state in $\langle K, \leq \rangle$, then by monotony (Lemma 1.22): A is valid in $\mathscr{K}$ iff A is valid in k_0.

Kripke-models $\mathscr{K} = \langle K, \leq, \Vdash \rangle$ can also be presented diagrammatically:

– We write states $k \in K$ as boxes $\boxed{k}$.

– If $k \leq k'$ holds for different states k, k', then we write

$$\begin{tikzcd} \boxed{k'} \\ \boxed{k} \arrow[u] \end{tikzcd}$$

We do not use arrows to indicate reflexivity or transitivity of $\leq$; however, these properties are always presupposed.

– Proposition letters that are valid in states $k \in K$ according to the valuation $\Vdash$ are written next to the boxes for the respective states; for example, if $k \Vdash p$ holds, then we write $\boxed{k} \Vdash p$.

Examples. (i) We show $\nVdash \neg\neg p \vee \neg p$. That is, we have to present a Kripke-model $\mathscr{K}$, such that $\mathscr{K} \nVdash \neg\neg p \vee \neg p$.

We consider the model $\mathscr{K}_1 = \langle K, \leq, \Vdash \rangle$ with

– $K = \{k_0, k_1, k_2\}$;

– $k_0 \leq k_1$ and $k_0 \leq k_2$ (besides $k_i \leq k_i$, which we do not note explicitly);

– $k_1 \Vdash p$.

Presented as a diagram, this model $\mathscr{K}_1$ looks as follows:

$$\begin{tikzcd} \boxed{k_1} \Vdash p & & \boxed{k_2} \\ & \boxed{k_0} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Since $k_1 \Vdash p$, we have $k_0 \nVdash \neg p$. For k_2 we have only $k_2 \geq k_2$, and $k_2 \nVdash p$; by Lemma 1.21 (i) it holds: $k_2 \Vdash \neg p$. Furthermore, $k_0 \nVdash \neg\neg p$.

Thus $k_0 \nVdash \neg\neg p \vee \neg p$ holds, and therefore $\mathscr{K}_1 \nVdash \neg\neg p \vee \neg p$. That is, $\mathscr{K}_1$ is a countermodel for $\neg\neg p \vee \neg p$; hence $\nVdash \neg\neg p \vee \neg p$.

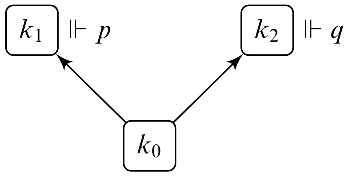
(ii) We show $\nVdash \neg\neg p \to p$. In the Kripke-model $\mathscr{K}_2$

$$\begin{tikzcd} \boxed{k_1} \Vdash p \\ \boxed{k_0} \arrow[u] \end{tikzcd}$$

we have that $k_0 \nVdash p$, and since $k_1 \Vdash p$ we have $k_0 \Vdash \neg\neg p$ (cp. Lemma 1.21 (ii)).

Hence $k_0 \nVdash \neg\neg p \to p$ holds, and therefore $\mathscr{K}_2 \nVdash \neg\neg p \to p$.

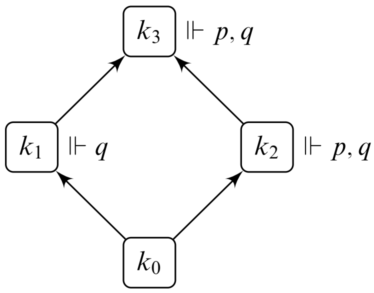
(iii) We show $\nVdash (p \to q) \vee (q \to p)$. A countermodel is $\mathscr{K}_3$



Since $k_1 \nVdash q$, we have $k_0 \nVdash p \to q$, and since $k_2 \nVdash p$, we have $k_0 \nVdash q \to p$.

Consequently, $k_0 \nVdash (p \to q) \vee (q \to p)$ holds, and thus $\mathscr{K}_3 \nVdash (p \to q) \vee (q \to p)$.

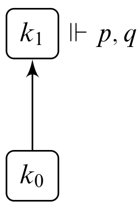
(iv) We show $\nVdash (p \to q) \to (\neg p \vee q)$. In the Kripke-model $\mathscr{K}_4$



we have $k_3 \Vdash p \to q$, and $k_1 \Vdash p \to q$ holds because of $k_1 \Vdash q$ and $k_3 \Vdash q$. Hence, with $k_0 \nVdash p$ and $k_2 \Vdash q$ also $k_0 \Vdash p \to q$ holds. However, $k_0 \nVdash \neg p$ holds due to $k_2 \Vdash p$, and since $k_0 \nVdash q$, we have $k_0 \nVdash \neg p \vee q$. Therefore $k_0 \nVdash (p \to q) \to (\neg p \vee q)$ holds, and thus $\mathscr{K}_4 \nVdash (p \to q) \to (\neg p \vee q)$.

The Kripke-model $\mathscr{K}_4$ illustrates that Kripke-models need not have the form of trees.

A smaller countermodel is $\mathscr{K}_5$:



Although $k_1 \Vdash (p \to q) \to (\neg p \vee q)$, we have $k_0 \Vdash p \to q$ but $k_0 \nVdash \neg p \vee q$. Thus $k_0 \nVdash (p \to q) \to (\neg p \vee q)$, and hence $\mathscr{K}_5 \nVdash (p \to q) \to (\neg p \vee q)$.

logical consequence

Definition 1.24 A formula A *is a logical consequence of* Γ (formally: $\Gamma \Vdash A$), if for every Kripke-model $\mathscr{K} = \langle K, \leq, \Vdash \rangle$ in every state $k \in K$, in which Γ holds, also A holds. That is:

$$\Gamma \Vdash A \;:\Longleftrightarrow\; \text{If } k \Vdash \Gamma, \text{ then } k \Vdash A, \text{ for every } \mathscr{K}.$$

(Where $k \Vdash \Gamma$ iff for all $B \in \Gamma$: $k \Vdash B$.)

Theorem 1.25 (Soundness and completeness) $\Gamma \vdash_{\mathrm{NI}} A \iff \Gamma \Vdash A$.

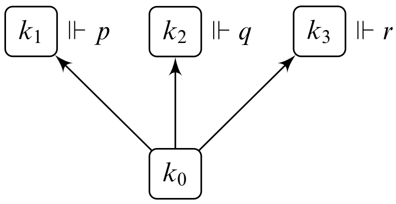
Proof. See van Dalen (2013, Ch. 5). (The result goes back to Kripke (1965), who proved soundness and completeness for an alternative calculus for intuitionistic logic.) QED

That a formula A is not derivable in NI ($\nvdash_{\mathrm{NI}} A$) can thus be shown by presenting a Kripke-countermodel for the formula (or for an instance of the formula). In this case $\nVdash A$ holds, and with soundness follows $\nvdash_{\mathrm{NI}} A$.

Example. We consider the claim

$$\neg p \to (q \vee r) \Vdash (\neg p \to q) \vee (\neg p \to r)$$

where the premiss $\neg p \to (q \vee r)$ is an instance of the premiss of Harrop's rule, and where the conclusion $(\neg p \to q) \vee (\neg p \to q)$ is an instance of the conclusion of this rule.
The Kripke-model $\mathscr{K}$



refutes this claim.

- In $\mathscr{K}$ the premiss holds: Since $k_1 \Vdash p$, also $k_0 \nVdash \neg p$, and thus $k_0 \Vdash \neg p \to (q \vee r)$. Hence by monotony $\neg p \to (q \vee r)$ holds in every state, i.e. $\mathscr{K} \Vdash \neg p \to (q \vee r)$.

  (For k_1, k_2 and k_3 one can also argue as follows: Since $k_1 \nVdash \neg p$, also $k_1 \Vdash \neg p \to (q \vee r)$. Since $k_2 \Vdash q$, we have $k_2 \Vdash q \vee r$, and therefore $k_2 \Vdash \neg p \to (q \vee r)$. Correspondingly for k_3, since $k_3 \Vdash r$.)

- But $\mathscr{K}$ is a countermodel of the conclusion: It is $k_3 \nVdash \neg p \to q$ and $k_2 \nVdash \neg p \to r$. Thus in k_0 neither $\neg p \to q$ nor $\neg p \to r$ holds, i.e. $k_0 \nVdash (\neg p \to q) \vee (\neg p \to r)$. Therefore $\mathscr{K} \nVdash (\neg p \to q) \vee (\neg p \to r)$.

Consequently

$$\neg A \to (B \vee C) \nVdash (\neg A \to B) \vee (\neg A \to C)$$

By soundness of NI we get

$$\neg A \to (B \vee C) \nvdash_{\mathrm{NI}} (\neg A \to B) \vee (\neg A \to C)$$

That is, Harrop's rule is not derivable in NI. This completes the proof of Theorem 1.10.

2 Dialogue semantics

Dialogues were proposed first by Lorenzen (1960, 1961) as an alternative foundation for constructive or intuitionistic logic. The general idea is that the logical constants are given an interpretation in certain game-theoretical terms. Dialogues are two-player games between a proponent and an opponent, where each of the two players can either attack claims made by the other player or defend their own claims. For example, an implication $A \to B$ is attacked by claiming A and defended by claiming B. This means that in order to have a winning strategy for $A \to B$, the proponent must be able to generate an argument for B depending on what the opponent can put forward in defense of A. The logical constant of implication has thus been given a certain game-theoretical or argumentative interpretation, and corresponding argumentative interpretations can be given for the other logical constants as well.

2.1 Dialogues and strategies

We define the concepts of argumentation form, dialogue and strategy, following the presentation of Felscher (1985, 2002) with slight deviations. We focus on dialogues for intuitionistic propositional logic.

2.1.1 Dialogues

We define our language, argumentation forms for logical constants and dialogues.

Definition 2.1 (i) The *language* consists of propositional *formulas* $A, B, C, \ldots$ that are constructed from *atomic formulas* (*atoms*) $q, r, s, \ldots$ with the *logical constants* $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication) and $\neg$ (negation). *language*

(ii) Furthermore, $\wedge_1$, $\wedge_2$ and $\vee$ are used as *special symbols*. *special symbols*

(iii) In addition, the *signatures* P ('proponent') and O ('opponent') are used. *signatures*

(iv) An *expression* e is either a formula or a special symbol. For each expression e there is a P-*signed expression* $P\,e$ and an O-*signed expression* $O\,e$. *expression*

(v) A signed expression is called *assertion* if the expression is a formula; it is called *symbolic attack* if the expression is a special symbol. X and Y, where $X \neq Y$, are used as variables for P and O. *assertion* *symbolic attack*

Definition 2.2 For each logical constant an *argumentation form* determines how a complex formula (having the respective constant in outermost position) that is asserted by X can be attacked by Y and how this attack can be defended (if possible) by X. The argumentation forms are as follows: *argumentation form*

conjunction $\wedge$:	assertion:	$X\ A_1 \wedge A_2$	
	attack:	$Y\ \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X\ A_i$	
disjunction $\vee$:	assertion:	$X\ A_1 \vee A_2$	
	attack:	$Y\ \vee$	
	defense:	$X\ A_i$	(X chooses $i = 1$ or $i = 2$)

implication $\to$:	assertion:	$X\ A \to B$
	attack:	$Y\ A$
	defense:	$X\ B$
negation $\neg$:	assertion:	$X\ \neg A$
	attack:	$Y\ A$
	defense:	*no defense*

Example. The following is a concrete instance of the argumentation form for implication:

$$
\begin{array}{l}
P\ \neg q \to (r \vee q) \\
O\ \neg q \\
P\ r \vee q
\end{array}
$$

argumentative interpretation

The argumentation forms provide what Felscher (2002, p. 127) calls an *argumentative interpretation* of the logical constants in the following sense:

(i) An argument on a conjunctive assertion made by X consists in Y choosing one conjunct of the assertion, and X continuing the argument with that chosen conjunct. In other words, the argumentative interpretation of conjunction is given by the reduction of the argument on a conjunctive assertion made by X to the argument on one of the conjuncts chosen by Y in the attack.

(ii) In an argument on a disjunctive assertion made by X, Y demands the continuation of the argument with any of the disjuncts. In other words, the argumentative interpretation of disjunction is given by the reduction of the argument on a disjunctive assertion made by X to the argument on one of the disjuncts chosen by X in the defense.

(iii) An argument on an implicative assertion made by X consists in Y stating the antecedent of the implication (whereby the antecedent functions as an assumption), and X continuing the argument with the succedent. Alternatively, X could continue with an attack on the assumed antecedent. In other words, the argumentative interpretation of implication is given by the reduction of the argument on an implicative assertion made by X to the argument on the succedent under the assumption of the antecedent.

(iv) An argument on a negative assertion $\neg A$ made by X consists in Y stating the assertion A, without X being able to continue the argument.

This argumentative interpretation of negation can be made clear by introducing the *falsum* $\bot$ as a constant which signifies absurdity (which is taken as a primitive notion). We can then define negation by implication and *falsum*: $\neg A := A \to \bot$. An argument on $\neg A$ is thus an argument on $A \to \bot$. However, X asserting $\bot$ would mean that Y could continue the argument with *any* assertion – assuming the principle of *ex falso quodlibet* to be applicable here. To avoid this, $\bot$ must not be asserted. Hence, an argument on $\neg A$ (i.e. on $A \to \bot$) can only continue with an argument on the assumption A, and cannot be reduced to an argument on $\bot$.

This is similar to the treatment of negation in constructive semantics, respectively in the BHK-interpretation of logical constants, as for example stated by Heyting (1971, p. 102): "[...] $\neg\mathfrak{p}$ can be asserted if and only if we possess a construction which from the supposition that a construction $\mathfrak{p}$ were carried out, leads to a contradiction." Where contradiction – or equivalently absurdity (here signified by $\bot$) – is usually considered to be a primitive notion.

Definition 2.3 (i) Let $\delta(n)$, for $n \geq 0$, be a signed expression and $\eta(n)$ a pair $[m, Z]$, for $0 \leq m < n$, where Z is either A (for 'attack') or D (for 'defense'), and where $\eta(0)$ is empty. Pairs $\langle \delta(n), \eta(n) \rangle$ are called *moves*. *move*

(ii) A move $\langle \delta(n), \eta(n) = [m, A] \rangle$ is called *attack move*, and a move $\langle \delta(n), \eta(n) = [m, D] \rangle$ is called *defense move*. *attack move* *defense move*

It is $\delta(n)$ a function mapping natural numbers $n \geq 0$ to signed expressions $X\ e$, and $\eta(n)$ is a function mapping natural numbers $n \geq 0$ to pairs $[m, Z]$. The numbers in the domain of $\delta(n)$ (resp. in the domain of $\eta(n)$) are called *positions*. *positions*

When talking about a move $\langle \delta(n), \eta(n) \rangle$, we write $\langle \delta(n) = X\ e, \eta(n) = [m, Z] \rangle$ to express that $\delta(n)$ has the value $X\ e$ for position n, and that $\eta(n)$ has the value $[m, Z]$ for position n. For example, $\langle \delta(n) = P\ A, \eta(n) = [m, D] \rangle$ denotes a defense move which is made by the proponent P at position n by asserting the formula A; this defense move refers to a move made at position m. A concrete move like $\langle \delta(4) = P \wedge_1, \eta(4) = [3, A] \rangle$ will also be written as

$$4. \quad P \wedge_1 \quad [3, A]$$

This is an attack move with symbolic attack $P \wedge_1$; it is made at position 4 and refers to a move made at position 3.

The notation $\langle \delta(n) = X\ e, \eta(n) = [m, Z] \rangle$ has the advantage that we can speak about a move $\langle X\ e, [m, Z] \rangle$ by including information about the position n at which this move is made.

Although moves are always pairs $\langle \delta(n), \eta(n) \rangle$, we will also refer to moves by giving only their $\delta(n)$-component, as long as it is clear from the context which move is meant, or if it is irrelevant whether the move is an attack or a defense, or if it is irrelevant to which position the move refers to. And instead of $\langle \delta(n) = X\ e, \eta(n) \rangle$ we will also speak of the move $X\ e$ made at position n. We will also speak simply about attacks and defenses in order to refer to attack moves and defense moves, respectively.

Definition 2.4 A *dialogue* is a finite or infinite sequence of moves $\langle \delta(n), \eta(n) \rangle$ (for $n = 0, 1, 2, \ldots$) satisfying the following conditions: *dialogue*

($D00$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

($D01$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

Definition 2.5 An attack $\langle \delta(n), \eta(n) = [m, A] \rangle$ at position n on an assertion at position m is called *open at position* k for $n < k$ if there is no position n' such that $n < n' \leq k$ and $\langle \delta(n'), \eta(n') = [n, D] \rangle$, that is, if there is no defense at or before position k to an attack at position n. *open attack*

Remark. Since there is no defense to an attack $\langle \delta(n) = Y\ A, \eta(n) = [m, A] \rangle$ on $\delta(m) = X\ \neg A$ for $m < n$, the attack at position n is open at all positions k for $n < k$.

2.1.2 DI-dialogues

We define DI-dialogues and strategies. With regard to the literature on dialogical logic, DI-dialogues can be considered to be the standard dialogues for intuitionistic propositional logic. The following definition of DI-dialogues is based on the definition of dialogues.

Definition 2.6 A *DI-dialogue* is a dialogue satisfying the following conditions (in addition to $(D00)$, $(D01)$ and $(D02)$): *DI-dialogue*

$(D10)$ If, for an atomic formula q, $\delta(n) = P\,q$, then there is an m such that $m < n$ and $\delta(m) = O\,q$.

That is, P may assert an atomic formula only if it has been asserted by O before.

$(D11)$ If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

$(D12)$ For every m there is at most one n such that $\eta(n) = [m, D]$.

That is, an attack may be defended at most once.

$(D13)$ If m is even, then there is at most one n such that $\eta(n) = [m, A]$.

That is, a P-signed formula may be attacked at most once.

A DI-dialogue beginning with $P\,A$ (i.e., $\delta(0) = P\,A$, where A is a complex formula) is called *DI-dialogue for the formula A*.

Remarks. (i) The objects defined by the conditions $(D00)$–$(D02)$ alone are what Felscher (1985, 2002) calls ‘dialogues’, and the objects defined by adding $(D10)$–$(D13)$ – which we call ‘DI-dialogues’ – are called ‘D-dialogues’ by him. Since here we are concerned with the objects defined by $(D00)$–$(D02)$ plus $(D10)$–$(D13)$, we simply speak of ‘dialogues’, omitting the specifier ‘DI’ as long as no confusion can arise. *(dialogues)*

(ii) The conditions $(D00)$–$(D13)$ are also called ‘structural rules’, ‘frame rules’ (‘Rahmenregeln’) or ‘special rules of the game’ (‘spezielle Spielregeln’) in the literature, and $(D10)$ is sometimes called ‘formal rule’. The argumentation forms are also called ‘particle rules’ (‘Partikelregeln’), ‘logical rules’ or ‘general rules of the game’ (‘allgemeine Spielregeln’).

We will stick to the notions ‘dialogue condition(s)’ (or just ‘condition(s)’) and ‘argumentation form(s)’. *(dialogue conditions)*

(iii) Proponent P and opponent O are not interchangeable due to the asymmetries between P and O introduced by $(D10)$ and $(D13)$. For atomic formulas q, the proponent move $\langle \delta(n) = P\,q, \eta(n) = [m, Z] \rangle$ is possible only after an opponent move $\langle \delta(m) = O\,q, \eta(m) = [k, Z] \rangle$ for $k < m < n$, and O can attack a P-signed formula only once, whereas P can attack O-signed formulas repeatedly.

These asymmetries are introduced by dialogue conditions only. The argumentation forms themselves (as given in Definition 2.2) are symmetric with respect to the two players P and O. That is, they are independent of whether the assertion is made by the proponent P or by the opponent O; they are thus player independent.

Definition 2.7 *P wins a dialogue for a formula A* if the dialogue is finite, begins with the move $P\,A$ and ends with a move of P such that O cannot make another move. *winning a dialogue*

Remark. A dialogue won by P ends with a move $\langle \delta(n) = P\,q, \eta(n) = [m, Z] \rangle$, where q is an atomic formula.

Example. A dialogue for the formula $(q \vee r) \rightarrow \neg\neg(q \vee r)$ is the following:

0.	$P\,(q \vee r) \rightarrow \neg\neg(q \vee r)$	
1.	$O\,q \vee r$	$[0, A]$
2.	$P\,\vee$	$[1, A]$
3.	$O\,q$	$[2, D]$
4.	$P\,\neg\neg(q \vee r)$	$[1, D]$
5.	$O\,\neg(q \vee r)$	$[4, A]$
6.	$P\,q \vee r$	$[5, A]$
7.	$O\,\vee$	$[6, A]$
8.	$P\,q$	$[7, D]$

The dialogue starts with the assertion of the formula $(q \vee r) \rightarrow \neg\neg(q \vee r)$ by the proponent P in the initial move at position 0. This initial move is attacked ($\eta(1) = [0, A]$) by the opponent O with the assertion of the antecedent $q \vee r$ ($\delta(1) = O\,q \vee r$) of the implication asserted by P at position 0. The attack is thus made according to the argumentation form for implication.
At position 2, the proponent does not proceed according to the argumentation form for implication by defending O's attack move with the assertion of the succedent $\neg\neg(q \vee r)$ of the attacked implication. Instead, the proponent makes the symbolic attack $P\,\vee$ on O's assertion $q \vee r$. This move is thus made according to the argumentation form for disjunction. The attack is defended by O with the assertion of the left disjunct q (alternatively, O could also have chosen the right disjunct r). The moves at positions 1–3 are an instance of the argumentation form for disjunction.
As q is an atomic formula, it cannot be attacked. At position 4, the proponent defends O's attack $O\,q \vee r$ by asserting the succedent $\neg\neg(q \vee r)$ of the attacked implication $(q \vee r) \rightarrow \neg\neg(q \vee r)$. The moves at positions 0, 1 and 4 are an instance of the argumentation form for implication.
The opponent now attacks $P\,\neg\neg(q \vee r)$ at position 5 by asserting $O\,\neg(q \vee r)$ according to the argumentation form for negation. By this argumentation form there is no defense for the attack. But the proponent can attack $O\,\neg(q \vee r)$ with the assertion $P\,q \vee r$. The moves at positions 4 and 5 are an instance of the argumentation form for negation, and the moves at positions 5 and 6 are another instance of that argumentation form.
Next O attacks $P\,q \vee r$ with the symbolic attack $O\,\vee$ according to the argumentation form for disjunction at position 7. Finally, this attack is defended by P's assertion of the left disjunct q. The moves at positions 6–8 are made according to the argumentation form for disjunction. Note that P cannot defend here by asserting the right disjunct r: the opponent has not asserted the atomic formula r before, hence such a move is prohibited by condition $(D10)$.
The proponent's move at position 8 is the last one. The opponent cannot attack q, since it is an atomic formula. Each other P-signed formula has been attacked by O, thus no more attack moves can be made by O due to condition $(D13)$, as these would be repetitions of attacks already made. And since each proponent attack that can be defended according to an argumentation form has already been defended by O, no more defense moves are

possible either, due to condition $(D12)$. The dialogue is finite, begins with the move $P\ (q \vee r) \to \neg\neg(q \vee r)$ and ends with a move of P such that O cannot make another move; the dialogue for the formula $(q \vee r) \to \neg\neg(q \vee r)$ is thus won by P.

2.1.3 Strategies

We next introduce dialogue trees and define strategies. We explain first what we call a path.

Definition 2.8 A *path* in a branch of a tree with root node n_0 is a sequence $n_0, n_1, \ldots, n_k$ of nodes for $k \geq 0$ where n_i and n_{i+1} are adjacent for $0 \leq i < k$. *path*

Definition 2.9 A *dialogue tree* is a tree whose branches contain as paths all possible dialogues for a given formula. *dialogue tree*

Example. Schematic example of a dialogue tree:

$$\begin{array}{llllll}
0. & & & P & & \\
1. & O & \vert & & & O \\
2. & P & \vert & P & \vert & P \\
3. & & & & \vert & O\ldots
\end{array}$$

At each odd position all possible moves for O have to be considered, and at each even position all possible moves for P have to be considered.

Remark. For a given formula A there is exactly one dialogue tree, if we consider trees to be equal modulo swapping of branches.

Definition 2.10 A *strategy* for a formula A is a subtree S of the dialogue tree for A such that *strategy*

(i) S does not branch at even positions,

(ii) S has as many nodes at odd positions as there are possible moves for O,

(iii) and all branches of S are dialogues for A won by P.

Example. Schematic example of a strategy:

$$\begin{array}{llllll}
0. & & & P & & \\
1. & O & \vert & & & O \\
2. & P & \vert & P & \vert & \cancel{P} \\
3. & & & & \vert & \cancel{O}\ldots
\end{array}$$

At each odd position all possible moves for O have to be considered (ii), but at each even position only *one* move for P has to be considered (i). The two remaining branches are dialogues won by P (iii).

Remarks. (i) In more game-theoretic terms, the strategies defined here could also be called *winning strategies for the player P*, and a corresponding definition could be given of *winning strategies for the player O*. For the dialogical treatment of logic undertaken here, only the first notion is needed, however. We can thus simply speak of *strategies*.

(ii) Strategies are finite for propositional formulas. All the branches in a strategy have finite length by definition, whereas dialogues that are not part of a strategy can be of infinite length. Dialogue trees are therefore infinite objects in general. As dialogue trees can be constructed breadth-first, of course, an existing strategy can always be found.

Formulas can have no, exactly one or more than one strategy.

Example. There is exactly one strategy for the formula $q \to \neg\neg q$:

$$
\begin{array}{lll}
0. & P\, q \to \neg\neg q & \\
1. & O\, q & [0, A] \\
2. & P\, \neg\neg q & [1, D] \\
3. & O\, \neg q & [2, A] \\
4. & P\, q & [1, A]
\end{array}
$$

The strategy contains only one branch.

Example. For the formula $(q \vee r) \to \neg\neg(q \vee r)$ there are the following three strategies, among others:

(i)

$$
\begin{array}{lll|ll}
0. & \multicolumn{4}{c}{P\, (q \vee r) \to \neg\neg(q \vee r)} \\
1. & \multicolumn{3}{c}{O\, q \vee r} & [0, A] \\
2. & \multicolumn{3}{c}{P\, \neg\neg(q \vee r)} & [1, D] \\
3. & \multicolumn{3}{c}{O\, \neg(q \vee r)} & [2, A] \\
4. & \multicolumn{3}{c}{P\, q \vee r} & [3, A] \\
5. & \multicolumn{3}{c}{O\, \vee} & [4, A] \\
6. & \multicolumn{3}{c}{P\, \vee} & [1, A] \\
7. & O\, q & [6, D] & O\, r & [6, D] \\
8. & P\, q & [5, D] & P\, r & [5, D]
\end{array}
$$

(ii)

$$
\begin{array}{lll|ll}
0. & \multicolumn{4}{c}{P\, (q \vee r) \to \neg\neg(q \vee r)} \\
1. & \multicolumn{3}{c}{O\, q \vee r} & [0, A] \\
2. & \multicolumn{3}{c}{P\, \neg\neg(q \vee r)} & [1, D] \\
3. & \multicolumn{3}{c}{O\, \neg(q \vee r)} & [2, A] \\
4. & \multicolumn{3}{c}{P\, \vee} & [1, A] \\
5. & O\, q & [4, D] & O\, r & [4, D] \\
6. & P\, q \vee r & [3, A] & P\, q \vee r & [3, A] \\
7. & O\, \vee & [6, A] & O\, \vee & [6, A] \\
8. & P\, q & [7, D] & P\, r & [7, D]
\end{array}
$$

(iii)

$$
\begin{array}{lll|ll}
0. & \multicolumn{4}{c}{P\, (q \vee r) \to \neg\neg(q \vee r)} \\
1. & \multicolumn{3}{c}{O\, q \vee r} & [0, A] \\
2. & \multicolumn{3}{c}{P\, \vee} & [1, A] \\
3. & O\, q & [2, D] & O\, r & [2, D] \\
4. & P\, \neg\neg(q \vee r) & [1, D] & P\, \neg\neg(q \vee r) & [1, D] \\
5. & O\, \neg(q \vee r) & [4, A] & O\, \neg(q \vee r) & [4, A] \\
6. & P\, q \vee r & [5, A] & P\, q \vee r & [5, A] \\
7. & O\, \vee & [6, A] & O\, \vee & [6, A] \\
8. & P\, q & [7, D] & P\, r & [7, D]
\end{array}
$$

There are more strategies for this formula than the three shown here, because the proponent can repeatedly attack formulas asserted by the opponent. For example, in strategy (iii) the proponent could at position 4 (in the left as well as in the right dialogue) repeat the attack $P\vee$ on $O\ q \vee r$. The subtrees below these attacks (in both dialogues) would have the same form as the subtree below position 2 in strategy (iii).

Example. There is no strategy for the formula $q \vee \neg q$, an instance of *tertium non datur*. The only possible dialogue is

0.	$P\ q \vee \neg q$	
1.	$O\ \vee$	$[0, A]$
2.	$P\ \neg q$	$[1, D]$
3.	$O\ q$	$[2, A]$

and P does not win.
There would be a strategy, if condition $(D12)$ were dropped for P. Then P could defend the attack $O\vee$ a second time by stating q, thereby winning the dialogue. Condition $(D11)$ does not have to be dropped because there are not more than one open attacks at position 3 (there is exactly one open attack at position 3; the attack $O\vee$ is not open there since it has already been defended at position 2).

2.2 Soundness and completeness

Definition 2.11 A formula A is called *dialogue-provable* (or *DI-dialogue-provable*) if there is a strategy for A. Notation: $\vdash_{\mathrm{DI}} A$. *dialogue-provable*

Remark. We speak of dialogue-*provable* formulas here, in accordance with Felscher (2002). Contrasting Gentzen's calculi with dialogues, Felscher (2002, p. 127) remarks:

> Gentzen's calculi of proofs are easily explained in that they represent the weakest consequence relation for which the provability interpretation is valid. The connection between dialogues and the argumentative interpretation of logical operations is [...] located on a different level: it is not the dialogues but the *strategies* for dialogues which will correspond to proofs. I thus formulate the *basic purpose* for the use of dialogues:
>
> (A_0) Logically provable assertions shall be those which, for *purely formal* reasons, can be upheld by a strategy covering every dialogue chosen by $[O]$.

However, the fact that we speak of *provability* in the context of dialogues (thus following Felscher) should not be misunderstood in a way that would imply that dialogues cannot be seen as a (formal) semantics (as opposed to considering dialogues only as a proof system or calculus).
Of course, such a misunderstanding could only arise if one's notion of semantics is limited to truth-conditional semantics, as opposed to proof-theoretic semantics (like the BHK-interpretation, or related justificationist, verificationist, pragmatist or falsificationist approaches in the tradition of Dummett and Prawitz) where the notion of proof or closely related notions are of central importance.
As the meaning of the logical constants is in some sense given by the argumentation forms in terms of how assertions containing the logical constants can be used in an

argumentation, dialogues might very well be seen as a semantics under the heading "meaning is use", and were indeed introduced for that purpose. This aspect can be emphasised by speaking of *(logical) validity* instead of dialogue-provability. *validity*

Theorem 2.12 (Soundness and completeness) *The dialogue-provable formulas are exactly the formulas provable in intuitionistic logic.*

This theorem has been shown (also for intuitionistic first-order logic) by Felscher (1985) by proving for Gentzen's sequent calculus LI (for intuitionistic first-order logic; see Gentzen, 1935) that every (first-order) strategy can be transformed into a proof in LI, and vice versa.

2.3 Addendum: Contraction in dialogues

In dialogues, the structural operations of thinning and contraction are only implicitly given by the dialogue conditions. This is comparable to natural deduction, where these structural operations are also only implicitly given, namely by how assumptions are discharged. Whereas in sequent calculus these operations are explicitly given as structural rules. That the structural operations are only implicitly given in dialogues can be seen as an advantage: we have argumentation forms only for the logical constants, and everything else is – in part implicitly – taken care of by the dialogue conditions.

Theorem 2.13 *In dialogues, the twofold use made by the proponent P of a formula A asserted by the opponent O corresponds to the structural operation of contraction, contracting A, A into A. The twofold use can consist either*

(1) in the twofold attack of a formula by the proponent P,

(2) in the twofold assertion by the proponent P of a formula asserted by the opponent O before,

or

(3) in an attack of a formula A by the proponent P together with the assertion of A by P.

That is, the twofold use can be of the following forms:

$$
(1)\quad \begin{array}{rll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,e & [k, A] \\ & \vdots & \\ m. & P\,e & [k, A] \end{array}
\qquad\qquad
(2)\quad \begin{array}{rll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,A & [i < l, Z] \\ & \vdots & \\ m. & P\,A & [j < m, Z] \end{array}
$$

$$
(3)\quad \begin{array}{rll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,e & [k, A] \\ & \vdots & \\ m. & P\,A & [i < m, Z] \end{array}
\qquad \textit{respectively} \qquad
\begin{array}{rll} k. & O\,A & [k-1, Z] \\ & \vdots & \\ l. & P\,A & [i < l, Z] \\ & \vdots & \\ m. & P\,e & [k, A] \end{array}
$$

Example. In the following two examples the twofold use made by P of an assertion made by O is of the form (1). The formulas $\neg(q \wedge \neg q)$ respectively $\neg\neg(q \vee \neg q)$ are not provable without a twofold attack on $q \wedge \neg q$ respectively $\neg(q \vee \neg q)$ by P, or without the corresponding discharge of two occurrences of the same assumption in the natural deduction derivations (where $\neg q := q \to \bot$), respectively:

(i)

0.	$P\,\neg(q \wedge \neg q)$	
1.	$O\,q \wedge \neg q$	$[0, A]$
2.	$P \wedge_1$	$[1, A]$
3.	$O\,q$	$[2, D]$
4.	$P \wedge_2$	$[1, A]$
5.	$O\,\neg q$	$[4, D]$
6.	$P\,q$	$[5, A]$

$$\dfrac{\dfrac{\dfrac{[q \wedge \neg q]^1}{\neg q}\,(\wedge\,\mathrm{E}) \qquad \dfrac{[q \wedge \neg q]^1}{q}\,(\wedge\,\mathrm{E})}{\bot}\,(\to \mathrm{E})}{\neg(q \wedge \neg q)}\,(\to \mathrm{I})^1$$

The twofold attack at positions 2 and 4 corresponds to the contraction of $q \wedge \neg q, q \wedge \neg q$ to $q \wedge \neg q$.

(ii)

0.	$P\,\neg\neg(q \vee \neg q)$	
1.	$O\,\neg(q \vee \neg q)$	$[0, A]$
2.	$P\,q \vee \neg q$	$[1, A]$
3.	$O \vee$	$[2, A]$
4.	$P\,\neg q$	$[3, D]$
5.	$O\,q$	$[4, A]$
6.	$P\,q \vee \neg q$	$[1, A]$
7.	$O \vee$	$[6, A]$
8.	$P\,q$	$[7, D]$

$$\dfrac{[\neg(q \vee \neg q)]^2 \qquad \dfrac{\dfrac{\dfrac{[\neg(q \vee \neg q)]^2 \qquad \dfrac{[q]^1}{q \vee \neg q}\,(\vee\,\mathrm{I})}{\bot}\,(\to \mathrm{E})}{\neg q}\,(\to \mathrm{I})^1}{q \vee \neg q}\,(\vee\,\mathrm{I})}{\dfrac{\bot}{\neg\neg(q \vee \neg q)}\,(\to \mathrm{I})^2}\,(\to \mathrm{E})$$

The twofold attack at positions 2 and 6 corresponds to the contraction of $\neg(q \vee \neg q), \neg(q \vee \neg q)$ to $\neg(q \vee \neg q)$.

2.4 Addendum: Classical dialogues

Although we are only concerned with intuitionistic logic, we point out here how dialogues for classical (propositional) logic relate to dialogues for intuitionistic (propositional) logic.

Theorem 2.14 *If the conditions* $(D11)$ *and* $(D12)$ *are restricted to apply only to* O *(and no more to* P*), then the formulas provable on the basis of the thus modified dialogues are exactly the formulas provable in classical logic.*

Definition 2.15 A *classical dialogue* is a dialogue where the conditions $(D11)$ and $(D12)$ do hold for O but not for P, that is, where conditions $(D11)$ and $(D12)$ are replaced by the following conditions $(D11^{+})$ and $(D12^{+})$, respectively: *classical dialogue*

$(D11^{+})$ If $\eta(p) = [n, D]$ for even n, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$.

That is, if at a position $p - 1$ there are more than one open attacks by P, then only the last of them may be defended by O at position p.

$(D12^{+})$ For every even m there is at most one n such that $\eta(n) = [m, D]$.

That is, an attack by P may be defended by O at most once.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for dialogues are directly carried over to the corresponding notions for classical dialogues.

The effects of replacing $(D11)$ and $(D12)$ by $(D11^{+})$ and $(D12^{+})$, respectively, are illustrated in the two following examples.

Example. There is a classical strategy for the formula $q \vee \neg q$:

0.	$P\, q \vee \neg q$	
1.	$O \vee$	$[0, A]$
2.	$P\, \neg q$	$[1, D]$
3.	$O\, q$	$[2, A]$
4.	$P\, q$	$[1, D]$

The last move is possible due to the replacement of condition $(D12)$ by condition $(D12^{+})$. In the presence of $(D12)$ this move is not possible, and there is thus no DI-strategy for (any instance of) *tertium non datur* (cf. the example on page 35).

Example. There is a classical strategy for the formula $\neg\neg q \to q$:

0.	$P\, \neg\neg q \to q$	
1.	$O\, \neg\neg q$	$[0, A]$
2.	$P\, \neg q$	$[1, A]$
3.	$O\, q$	$[2, A]$
4.	$P\, q$	$[1, D]$

The last move is possible due to the replacement of condition $(D11)$ by condition $(D11^{+})$. In the presence of $(D11)$ this move is not possible, and there is thus no DI-strategy for (any instance of) double negation elimination.

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