Inference Rules and the Meaning of the Logical Constants

An Elaboration of Dummett's Notions of Proof-Theoretic Validity

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To the memory of Michael Dummett (1925–2011).

May his works and social activism
inspire many generations to come.

Preface

When I first started to study logic, the idea of validity as preservation of truth from premisses to conclusion always baffled me. How is it possible to explain validity in terms of preservation of truth when there are valid arguments where all sentences, premisses and conclusion, are false? What is being preserved in this case? "Well, you see", somebody would try to explain, "in this case, *if* the premisses *were* true, then the conclusion *would be* true." "This is just a roundabout way of *assuming* that the argument is valid," I would usually reply. "More precisely, how could I ever evaluate that criterion without already presuposing that the argument is (in)valid?" It seemed to me that the truth preservation effect was a *consequence* of validity, not an *explanation* of it.

Later I learned that this idea of truth preservation is in fact a somewhat misleading oversimplification of the classical theory of validity. Still, I was not completely satisfied. In the classical explanation of validity, the meaning of the propositional logical constants, as explained through their truth tables, is based on the doctrine that the pertinent trait of sentences, their semantic value, consists in being either true or false (bivalence).

Although bivalence is arguably a compelling and unproblematic doctrine in many contexts, our deductive practice does not seem to be at all restricted to those contexts. For instance, we usually perform refutations in contexts where we do not care whether premisses and conclusion are true: we merely try to show our interlocutor that, if one rejects the conclusion, one must reject at least one of the premisses. Deductive arguments also appear in fiction (detective novels, fantasy novels, philosophical novels and what have you). The sentences therein are meant to be neither true nor false in the technical sense of logical theory. What unfolds in those contexts are deductions nonetheless and a satisfactory notion of validity should account for them.

From what I can gather, the orthodoxy would have us believe that the validity of arguments in these contexts are subordinate to the validity of the same argumentative forms in truth preservation, or, to be more exact, bivalent scenarios. I still contend however that ambivalent deduction, that is, deduction indifferent to alethic repercussions, is deduction enough.

Despite its astounding prominence, the classical theory of validity places itself somewhat removed from practice. Now, idealization in itself is not a bad thing, but too much of it can leave us with a stagnant and dogmatic theory. The question is, of course, can we do better?

Any alternative must strike a better balance between adequacy to practice and theoretical simplicity and robustness. In terms of adequacy to practice, the classical theory sure leaves a lot to be desired. For evidence of its inadequacy, one need look no further than to the infamous paradoxes of material implication. Furthermore, because of its prominence, the perceived simplicity and robustness of the classical theory can be easily overestimated.

A logician, especially if she is not terribly concerned with understanding and explanation (either in general or for the purpose of some task at hand), may be perfectly content with a coarse extensional match between theory and practice, as long as it enables her to pursue formal and theoretical objectives. Nothing wrong with that. In fact, it explains why homophonic clausal definitions—for instance, "$A \to B$ is true if, and only if, *if* A is true, *then* B is true", where "$\to$" is supposed to stand for "if…then…"—are often used despite being poor explanations, since they *presuppose* understanding of the classical logical constants on the metalevel (an actual explanation would need to resort to the truth tables at some point).

Expressing discontent and advancing critiques to the classical theory is all well and good. Proposing satisfactory fixes and alternatives is much harder. Particularly, finding a better working balance between theory and practice. As a philosopher, I would gladly accept some loss in simplicity for more explanatory power and perspicuity. But as a logician, I would certainly rejoice in a systematic theory that enables formal pursuits and rigorous development.

A surprising outlet for my frustrations appeared when I read Dummett's *The Logical Basis of Metaphysics* as an undergraduate student. The book outlined the rudiments of a far reaching philosophical programme. The programme resonated strongly with me. The project of explaining meaning in terms of language use seemed particularly refreshing and undogmatic. I found, however, many things with which I didn't agree; I found many arguments wanting. In particular, I felt that the focus on knowledge, assertion and verification was misplaced. Nevertheless, while reading the book, the mixture of enthusiastically shared prospects and subtle philosophical disagreement provided some of the most enjoyable moments I had with a philosophical work.

I would like this text to be considered as a contribution to Dummett's Programme. It concerns the small, but, I believe, fundamental aspect of the programme that deals with logical validity.

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This dissertation incorporates some material, developed during my doctoral research, which have already appeared elsewhere:

- *Investigations on Proof-Theoretic Semantics.* Revised and expanded edition of my master's thesis (originally submitted in Portuguese).
  Available at `https://sdf.org/~hhebert/masters.pdf`.

- *On Dummett's Verificationist Justification Procedure.* Paper published in collaboration with Wagner de Campos Sanz in "Synthese", volume 193, issue 8, pages 2539–2559, 2016.
  Available at `https://philarchive.org/archive/SANODV`.

- *Revisiting Dummett's Proof-Theoretic Justification Procedures.* Contribution to "The Logica Yearbook 2016", edited by Pavel Arazim & Tomáš Lávička, College Publications, pages 141–155, 2017.
  Available at `https://philarchive.org/archive/OLIRDP`.

- *On Dummett's Pragmatist Justification Procedure.* Manuscript submitted for publication. Available at `https://arxiv.org/abs/1701.03380`.

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Contents

1 Meaning and Use

The idea that meaning relates to use is simple. Yet, its success is highly dependent on unravelling the details and achieving an adequate understanding of its consequences. A straightforward, but ultimately misguided, approach to explore the connection between meaning and use would be to *identify* the meaning of an expression with its use, any use. This naive approach faces two challenges: first, it is not able to systematically distinguish between correct and incorrect use; second, it does not distinguish between essential (or canonical) and inessential (or derivative) uses.

Instead of identifying meaning with an amalgam of particular uses, a sophisticated approach would associate meaning with *general rules* governing the use of expressions in the language. For natural languages, the formulation of these rules may seem to be a very daunting endeavour since, for the most part, they cannot be easily extracted from explicit usage patterns. Nonetheless, intersubjective standards for the correct use of expressions suggest that their use is indeed regimented by rules. These rules implicitly permeates our linguistic practices and are an important component in our ability to speak our languages.

Although correct and incorrect uses can be reliably recognised as such by competent speakers, it is not a straightforward task to make explicit what exactly are the general rules governing the use of many expressions in natural languages. But, the complexities and subtleties of natural language notwithstanding, the question seems to be more palatable when restricted to the logical constants. The syncategorematicity, universality and neutrality commonly attributed to the logical constants mean that they represent an especially circumscribed paradigmatic case. In particular, the logical constants lend themselves very well to a compositional treatment.

Compositionality is important because it enables a recursive explanation of the meaning of complex sentences on the basis of the meaning of their components. The recursive explanation is usually formulated in terms of some central semantic notion like truth or assertability. Whether based on truth or assertability, the generally accepted recipe is to describe how each logical constant behaves with respect to this central semantic value: how the semantic value of complex sentences are determined in terms of the semantic value of their constituents. Once the meaning of the logical constants are specified, validity can be easily defined in agreement to the idea of transmission, or preservation, of a designated semantic value.

Naturally, such an explanation of validity rests upon background philosophical doctrines regarding meaning and language. In classical theories, for instance, there are often doctrines relating meaning and truth (for example, bivalance), as well as doctrines about predication and denotation (in the case of predicate logic), among others. Dummett (1991) called these background philosophical doctrines, when systematically arranged, a *theory of meaning.*

Like many of his contemporaries who were heavily influenced by Frege, Dummett maintained that theories of meaning provided a bridge between logic and the philosophy of language. He further believed that this connection could be exploited in order to strategically approach the solution to a broad class of metaphysical disputes.

Persuaded by Wittgenstein ([1953] 2003), Dummett favoured theories of meaning more susceptible to linguistic practices and to the social character of language use. This led to the rejection of core doctrines of classical theories. Walking the bridge from the theory of meaning to logic, and from logic to metaphysics, Dummett sided with the intuitionist in matters of logic and mathematics, and appeared generally resistant to realism in other metaphysical disputes.

In this work, however, I am only concerned with the viability of a relatively small development of Dummett's journey: an explanation of validity on the basis of inference rules taken as expressing the deductive use of the logical constants. This endeavour ideally culminates in an inferentialist definition of validity. The approach that I favour is developed in agreement as well as in opposition to Dummett. Before focusing on the issue of validity in the next chapters, I discuss the philosophical background, arguments and doctrines that originate from the idea of "meaning as use" and revisit some of the challenges it poses to classical theories of meaning.

1.1 The social character of meaning

1.1.1 Manifestability

Theories of meaning are concerned with general questions about meaning and language. They are the philosophical background to the semantics of particular, natural or artificial, languages. For instance, they can prescribe what a meaning explanation should look like. They can give an account of how meaning relates to linguistic practice and of what constitutes our ability to speak the language. The breadth and reach of a theory of meaning, of course, depends on characteristics of the target language and the range of linguistic phenomena the semantics purports to explain.

For instance, if interested solely in mathematical sentences, the main question for a theory of meaning could be "What is the meaning of a mathematical sentence?".

There can be many answers. Two general approaches are particularly prominent in the philosophy of mathematics. First, the meaning of a mathematical sentence can be explained along the lines commonly associated with the logicists: a mathematical sentence is a description of certain immaterial and atemporal objects. These objects have a reality all of themselves, independent of human cognition, practice or existence.

Another approach, which is in clear opposition to the first one, holds that the meaning of a mathematical sentence has to be explained in terms of its use. And, although they can be used in many different contexts and situations, the chief use of mathematical sentences are in mathematical calculations and proofs. This approach can be loosely ascribed to the early intuitionists, as long as mathematical calculations and proofs are considered to be primiraly mental constructions.

Dummett (1975a) indeed suggested a more linguistic interpretation of the intuitionistic position, one that renounces the mentalism and solipsism associated with Brouwer. Under this new linguistic guise, the intuitionist can challenge the logicist on, at least, two grounds. First, from the perspective of the theory of meaning, the logicists allow for the possibility of a widespread and, worst of all, undetectable communicative failure. Second, the intuitionistic position can give a more clear account of the social character of the language, particularly of the practices of teaching and learning the language.

More precisely, suppose that the meaning of a mathematical sentence is indeed somehow tributary to the mathematical reality that it describes. Granted that the mathematical reality is objective and immaterial (not given to the senses), our grasp of the meaning of a particular mathematical sentence should consist in some private mental content, possibly obtained through intellectual intuition (or something of the sort). Now, the problem is how can we be sure that we associate the same content, that is, the same meaning, to the same sentence? In other words, how can a speaker of the language be sure that she associates to a given mathematical term, say "2^3", the same meaning that some other speaker? It seems that, if all there is to the meaning of a mathematical sentence is a correspondence to an immaterial mathematical reality, one speaker can never be sure to understand such a sentence the same way another speaker does.

For very much the same reasons, a teacher of the language cannot be sure to have teached it correctly. The learner also cannot be sure to have learned it correctly. It is very difficult for any theory of meaning completely indifferent to linguistic practices to give a satisfactory account of the teaching and learning of mathematics.

Yet why should the teaching and learning of mathematics be of any concern to a theory of meaning for mathematical sentences? What does it mean for the logiscist to hold that there are immaterial and atempotal mathematical objects inhabiting a reality independent of ourselves? What are the consequences of an insistence on incorporating use as an essential element of meaning? How would the intuitionistic theory of meaning differ from the logicist theory of meaning?

Similar to his approach to the intuitionistic metaphysics of mental constructions, Dummett reinterprets the logicist metaphysics of mathematical objects in terms of the theory of meaning. According to him, the core of the dispute can be rendered as a dispute around the legitimacy of the principle of bivalence when applied to mathematical sentences. The rejection of the principle of bivalence is believed to follow from a commitment to language use. That is, a theory of meaning based on use cannot admit the principle of bivalence as an universally legitimate principle. Thus, the intuitionistic deference to linguistic practice can be seen in the form of a general requirement placed on the theory of meaning: *the requirement of manifestability.* This requirement is designed to challenge any conceptions of meaning that are based upon some transcendent notion of truth which is independent from human knowledge and practices.

Manifestability In a theory of meaning, the explanation of the meaning of an expression must be formulated exclusively in terms of notions and distinctions which are completely manifestable in the linguistic practices of speakers.

The principle of bivalance is incompatible with the requirement of manifestability. There are some features of our language that enable the construction of sentences for which bivalance is arguably not manifestable. Some of them are (Dummett 1991, p. 315):

- our use of unbounded quantification over infinite totalities
- our use of the subjunctive conditional
- our capacity to refer to inaccessible regions of space-time, such as the past and the spatially remote

These features enable the composition of sentences whose truth is potentially transcendent, that is, sentences for which ascertaining their truth or falsity is either accidentally or, perhaps unbeknownst to us, necessarily out of reach. For instance, it may be impossible to appraise the truth of the sentence

An odd number of people attended Hypatia's lectures in 392 AD.

The sentence is meaningful, and any theory of meaning that purports to respect the requirement of manifestability must reject the principle of bivalance for these types of sentences. In other words, depending on the domain, the requirement of manifestability casts the principle of bivalance as a highly objectionable doctrine. Now, the salient questions are: Why should manifestability be a requirement at all? Why should a theory of meaning be susceptible to linguistic practice, give an account of the teaching and learning of the language, be concerned with verification instead of truth?

These are methodological questions. The answers are informed by what objectives a theory of meaning is called to fulfil. A classical logician may be perfectly content with a somewhat idealised theory of meaning and, consequently, an idealised concept of validity. She may concede the points of her opponents about the shortcomings of classical theories when measured against certain linguistic practices. She may even abstain from the realist metaphysics of immaterial realities and transcendent truth. She can still contend that classical theories are as good as they come, and that, through idealisation, one attains generality and simplicity which, in the wider scheme, are worthwhile trade-offs.

Indeed, critiques are certainly more powerful when paired with strong alternatives. The particular alternative pursued by Dummett (1975a) revolves around the "replacement of the notion of truth, as the central notion of the theory of meaning, by that of verification." In this context, the notion of verification is understood in a broad sense. In mathematical scenarios, for instance, proofs would count as verifications. In contingent scenarios, verifications could be empiric observations. In argumentative scenarios generally, we would consider justifications. The overall strategy is to move from a framework based on transcendental truth to one based on knowledge and the justifications we offer each other for establishing knowledge.

1.1.2 Assertion

Among the speech acts we use, assertions are the ones most relevant when it comes to knowledge in general and logic in particular. In verificationist theories of meaning, assertability is brought to the foreground. Thus, instead of the concept of truth, familiar from classical theories of meaning, a theory of meaning committed to use would take assertability as its central semantic concept.

In contrast with the abstract concept of proposition, assertions, as speech acts, carry with them an implicit commitment, by the person making the assertion, to stand for its correctness. In other words, this person can be challenged to offer justifications for the assertion and thus to make explicit the grounds upon which her or his knowledge of its correctness rests.

Notice that the conditions to correctly assert a sentence and the conditions for it to be true (in a classical sense) may be different. For instance, it is expected that, if a mathematician asserts, as opposed to merely states, Goldbach's conjecture, then she should be able to provide a proof. On the other hand, the truth of Goldbach's conjecture, supposing it is indeed true, does not depend on any *particular mathematician* being able to provide a proof. Actually, being a conjecture of elementary number theory, the intuition is that the correctness of the conjecture is settled even if *no mathematician* is able to come up with either a proof or a refutation. A verificationist can very well concede the point whilst denying that the *meaning* of the conjecture hangs on this bivalent state of affairs. According to the verificationist, the meaning of Goldbach's conjecture does not rely on a proof or refutation ever being found, but rather consists in our capacity to recognise either one of them once they are presented to us.

While a speech act of assertion itself is performed by an individual speaker in face of interlocutors, the assertability conditions invoked by the verificationist are considered to be the intersubjective product of our practice of offering, recognising and committing to justifications. It is thought to be the common denominator exchanged during the teaching and learning of the language which enables us to understand sentences like Goldbach's conjecture even though its correctness may lie outside our capacity to ascertain.

The verificationist framework for the theory of meaning is thus heavily epistemic in nature. The change from truth to knowledge, however, comes with its own complications. For instance, it requires great care not to confuse verifiability with actual verification (Prawitz 1998, § 4), a problem already familiar to the logical positivists (Creath 2017, § 4.1). Another challenge for the verificationist is to give an account of the content of sentences that can be shared between distinct speech acts (for example, supposition and denial) without falling back into a realist abstract notion of proposition.

Furthermore, in contrast with bare sentence content, assertion and knowledge relate, first and foremost, to individuals. We can say that a person knows something, but another does not; a person asserts something, but another denies it. It could be argued that this feature is at odds with the task of supporting an intersubjective theory of meaning.

1.2 Logic and mathematics

Together with a semantics for logic based on the "meaning as use" approach to the theory of meaning, another motivation of Dummett's Programme is to advance an argument for the adoption of intuitionistic over classical logic. This motivation stems from constructivist views on the philosophy of mathematics and seeks to gain support for constructive mathematics by replacing classical logic with constructive logic. The novelty, in comparison with early intuitionists and constructivists, is to approach these matters through the theory of meaning.

In outline, Dummett's Programme begins at the level of the philosophy of language by advancing arguments against classical theories of meaning. Then, an alternative conception of meaning based on use is proposed. The programme culminates in the development of a semantics for logic. Furthermore, the expectation is that the semantics will avail intuitionistic logic over classical logic and, finally, *through logic*, will settle the controversy in the philosophy of mathematics.

Dummett ([1963] 1978) maintains that the metaphysical dispute between classical and intuitionistic mathematics is part of a wide range of metaphysical disputes between two general opposing camps: realism and anti-realism. In his interpretation of this class of metaphysical disputes, the difference between the opposing camps boils down to the question of what is the correct theory of meaning for the relevant class of sentences and, in particular, whether the principle of bivalence applies.

The connection between the theory of meaning and metaphysics has often been criticized on general grounds, for instance, by Pagin (1998) and Devitt (1983). Notwithstanding, the constructive aspirations of Dummett's Programme can still fail on their own terms if inferentialist notions of validity fail to single out intuitionistic logic.

Indeed, many aspects of Dummett's Programme depend on the success of its inferentialist notions of validity. As Dummett (1991, chapter 10) himself observes, not only the mere possibility of a theory of meaning based on use, but also its capacity to criticise and maybe reform accepted linguistic usage faces important threats, for example, from semantic holism. Therefore, as a paradigmatic case, a successful try out of the "meaning as use" idea in the form of an inferentialist notion of validity would dispel doubts about its tenability and provide a mature competitor to classical theories of validity.

While my primary concern is this logical core of Dummett's Programme, the influence of intuitionism and its constructive heritage warrants an examination of some conceptual issues therein.

1.2.1 Intuitionism

The early 20th century witnessed a vigorous debate around the foundations of mathematics. As the logicism of Frege, Russell and Whitehead fell prey to paradoxes, two distinct philosophies came out as alternatives to logicism: Hilbert's formalism and Brouwer's intuitionism.

On the one hand, Hilbert's foundational program aimed to show the consistency of mathematics by means of finitistic methods. If carried out, Hilbert's consistency proof was believed to provide an indirect foundation for classical mathematics where the more direct approach of the logicists have failed. On the other hand, Brouwer's philosophy rejected any need for foundations: he characterized mathematics as a free product of the mathematician's mental constructions.

At that time, the intuitionistic critique of classical mathematics resonated well in the uneasy context of the paradoxes. Their philosophies remained quite distinct, despite the fact that, on a practical level, formalists (in their metamathematics) as well as intuitionists tried to restrict the principles of reasoning used (if compared to logicists). As Dummett (2000) observes:

> Intuitionism took the fact that classical mathematics appeared to stand in need of justification, not as a challenge to construct such a justification, direct or indirect, but as a sign that something was amiss with classical mathematics. From an intuitionistic standpoint, mathematics, when correctly carried on, would not need any justification from without, a buttress from the side or a foundation from below: it would wear its own justification on its face.

As it turned out, Brouwer's view of mathematics as mentally constructed would have drastic consequences if imposed on the mathematical practice of his day. One consequence would be the rejection of actual infinity, a concept that has become widely accepted, especially after Cantor's work on set theory. In this respect, Brouwer was expressing mathematical intuitions which go back, at least, to Gauss (1900, Letter to Heinrich Christian Schumacher, 18 July 1831). However, many mathematicians, including Hilbert (1926), were not pleased with the possibility of being expelled from "Cantor's paradise".

In addition to the foundational investigations into mathematics, Brouwer (1908) was also suspicious of logic. He understood clearly that, if the traditionally accepted principles of logic were indeed universally applicable, his views on how mathematics ought to be carried out could not stand unharmed. Thus, he believed that the most elementary constructions of mathematics are not in need of any foundation, logical or otherwise. Rather, he maintained that the logical theories advanced by the logicists as a purported foundation for mathematics in fact *presupposed* elementary mathematical techniques. Brouwer (1908) wrote about the principles of logic as being "unreliable". He was undoubtedly attacking the canons of reasoning associated with the predominant classical logic. But he was generally supportive of Heyting's efforts to propose a formal system capturing the intuitionistic canons of reasoning (van Atten 2017, § 2).

1.2.2 The principle of excluded middle

As already remarked, my main concern is *logic* rather than mathematics. Brouwer's philosophy of mathematics is nonetheless relevant because his reflections on the nature of mathematics led to the rejection of what seems to be a purely logical principle: the principle of excluded middle. Indeed, he blamed the paradoxes on the careless use of the the principle of excluded middle by mathematicians, especially when reasoning about potentially infinite mathematical series.

> It was Brouwer who first discovered an object which actually requires a different form of logic, namely the mental mathematical construction. The reason is that in mathematics from the very beginning we deal with the infinite, whereas ordinary logic is made for reasoning about finite collections. (Heyting 1956, chapter 1)

But why should the validity of a *logical* principle depend on whether the universe of discourse is finite or infinite? In his renowned dialogue, Heyting (1956, chapter 1) expressed the view that perhaps the principle of the excluded middle should not be considered a logical principle at all since it embodies an unjustified metaphysical assumption: that mathematical objects exist independently of our knowledge of them, i. e. independently of being constructed. In other words, the classical canons of reasoning incorporate extraneous methodological doctrines which, although legitimate and useful in certain domains, do not enjoy the universality required of logical principles. Therefore, the intuitionistic objection can be seen not only as a contribution to the debate in the philosophy of mathematics but also to the philosophy of logic. This interpretation informs, to a large extent, Dummett's Programme discussed before.

Realist metaphysical doctrines notwithstanding, the intuitionistic contention can also be placed on the "no ignorabimus" claim for the general solvability of mathematical problems (Hilbert 1900). In fact, Brouwer's method of weak counterexamples rests on the possibility that meaningful mathematical problems could remain unsolved (Troelstra and van Dalen 1988, § 1.3). These counterexamples point out applications of the principle of excluded middle which, under an intuitionistic interpretation, would lead to the conclusion that we either possess, or would certainly one day possess, proofs for any mathematical conjecture.

But, however (un)successful in challenging the classic metaphysical assumptions about mathematics, the intuitionist seems to advance a metaphysical picture of his own: that of mathematics as a product of mental constructions. The early intuitionistic notion of a mental mathematical construction was in desperate need of more careful and detailed explanation. Not at all by chance, there emerged different proposals for how mathematics should be developed intuitionistically. One interesting example thereof is the negationless mathematics of Griss (1946).

A related issue is whether an intuitionist should accept as justification constructions that are possible to effect in principle but were not actually carried out, and, perhaps *cannot* be actually carried out. For instance, consider the sentence

The sequence of digits "49027365293754" occurs in the decimal expansion of π somewhere before the $10^{10^{10}}$ decimal place.

Could we, from an intuitionistic point of view, correctly assert that it is either provable or refutable? The intuitionist would probably agree that to tie mathematics up to *completely effected* constructions made by actual persons, even the entire human race (past, present and future), is to make mathematics depend too much on casual contingent facts. In this case, the availability of a general method, say Archimedes method for calculating the decimal expansion of π, should suffice for the intuitionist.

Yet, the intuitionist seems to be vulnerable now to the same kind of critique that he or she has been directing against the classical mathematician (Dummett 1975b). So, why stop at intuitionism? Why not embrace some sort of finitism? After all, finitism seems to agree perfectly with the general constructive position that mathematical sentences express the realization of a mental construction (Bernays 1935). Even barring those critiques, given that the notion of mental mathematical construction is so open to interpretation, why not embrace the negationless mathematics of Griss (1946)?

These are important philosophical questions concerning the objectivity and nature of mathematics. They reveal the metaphysical nature of early intuitionistic positions and that, despite very reasonable claims to its significance for the philosophy of logic, its contributions are not easy to sort out.

1.2.3 The BHK interpretation

Heyting (1930) codified the principles of reasoning acceptable to the intuitionists. His first formulation was an axiomatic system. There were also attempts by Kolmogoroff (1932) to interpret the intuitionistic understanding of the logical constants in terms of solutions for problems. Finally, Heyting (1956, § 7.1.1) gave what is widely recognised as the definitive formulation, known as the BHK interpretation of the logical constants.[1]

$p \wedge q$ can be asserted if and only if both p and q can be asserted.

$p \vee q$ can be asserted if and only if at least one of the propositions p and q can be asserted.

$p \to q$ can be asserted, if and only if we possess a construction r, which, joined to any construction proving p (supposing that the latter be effected), would automatically effect a construction proving q.

$\neg p$ can be asserted if and only if we possess a construction which from the supposition that a construction that proves p were carried out, leads to a contradiction.

Sometimes, Heyting's clauses are adapted. Troelstra and van Dalen (1988, § 1.3.1), for example, define negation (section 2.2.4) in terms of implication and contradiction, also called absurdity ($\bot$), and then add a clause to the effect that there is no proof of absurdity. They also replace Heyting's notion of assertion by a notion of proof, thus tacitly assuming the thesis that we can correctly assert a proposition, if, and only if, we have a proof of it.

There is no doubt that the BHK interpretation indeed offers some insight into the intuitionistic meaning of the logical constants. For instance, using BHK and some intuitive notion of proof, we could argue for the validity of some logical laws, like double negation introduction.

$A \to \neg\neg A$ There is no proof of $\bot$. So, once in possesion of a proof of A, it is impossible to have a proof of $\neg A$, that is of $A \to \bot$, since that would in fact yield a proof of $\bot$ by *modus ponens*.

1. Heyting (1956, § 7.1.1) insists that the clauses apply to actual propositions and he uses gothic letters to distinguish between propositions and propositional variables. In this context, generality is achieved by an additional clause: "A logical formula with propositional variables, say $\mathfrak{U}(p, q, \ldots)$, can be asserted, if and only if $\mathfrak{U}(\mathfrak{p}, \mathfrak{q}, \ldots)$ can be asserted for arbitrary propositions $\mathfrak{p}, \mathfrak{q}, \ldots$; that is, if we possess a method of construction which by specialization yields the construction demanded by $\mathfrak{U}(\mathfrak{p}, \mathfrak{q}, \ldots)$".

Early inferentialist notions of validity have been tightly associated with constructive logic and mathematics. Thus, in order to fulfil the mission of providing a constructively acceptable semantics for intutionistic logic, inferentialist conceptions of validity have usually been developed under the shadow of the BHK interpretation (chapter 3). Dummett (2000, § 7.2) is no exception:

> There is no doubt, however, that the standard intuitive explanations of the logical constants [BHK] determine their intended intuitionistic meanings, so that anything which can be accepted as the correct semantics for intuitionistic logic must be shown either to incorporate them or, at least, to yield them under suitable supplementary assumptions.

In contrast, I favour a perspective which is largely derived from Dummett's, but rests instead on inference rules in natural deduction, giving no primacy to BHK whatever (chapter 4). Admittedly, there are some similarities between the BHK clauses and the introduction rules of natural deduction (chapter 2). These similarities, however, are deceptive. The differences are significant enough to advise that natural deduction and BHK be kept safely apart. At least two differences between the BHK interpretation and natural deduction are:

- the BHK clauses are formulated in terms of *proofs* (assertions) while natural deduction rules expresses the conditions for infering sentences on the basis of *assumptions*, i. e. hypotheses
- the BHK clause for implication is substantially different from its introduction rule in natural deduction[2]

Notwithstanding its plausibility as an explanation of the meaning attributed to the logical constants by intuitionists, the BHK clauses face many problems, both technical and conceptual, if called to act as semantic clauses for a systematic theory of meaning. Certainly, in mathematical contexts, is very hard to deny that we are only entitled to assert a sentence when we have a proof for it. Still, proof cannot be all there is to the meaning of mathematical sentences. Otherwise, what is the meaning of a mathematical conjecture?[3] In contrast, an

2. Prawitz (1971, § 2.1.1) also notes that the meaning of the introduction rule for implication is more strict than that of the corresponding BHK clause and concludes: "There is thus not a complete agreement but a close correspondence between the constructive meaning of the constants and the introduction rules."

3. Some mathematicians might say that the wisest thing to do with a conjecture is to remain silent about it until we have something relevant to say, that is, until we have a proof. Historically, then, it seems that mathematical practice has not always been on the wisest path.

approach based on deductions from assumptions does not face the same problem: conjectures, whilst not established, can still be used in our deductions as assumptions in order to extract consequences.

Yet another problem relates to the meaning of the absurdity constant: as long as meaning is defined in terms of proof, or conditional proof, what is the meaning of the absurdity constant which, by definition, has no proof? Indeed, it was because of this problem that Griss (1946) abandoned negation altogether. In natural deduction, on the other hand, the absurdity constant is only expected to be used in subordinate deductions from assumptions. In this regard, an approach based on natural deduction also seems to indicate a way out of the dilemma.

The BHK interpretation is primarily concerned with proofs as mental mathematical constructions. Gentzen (1934), on the other hand, had a very different motivation:[4]

> My motivation was: The formalization of logical deduction, particularly as developed by Frege, Russell and Hilbert, is somewhat removed from deductive practices as they are carried out in mathematical proofs. Considerable formal advantages are gained in return. Now, I would like to finally devise a formalism that comes as near as possible to actual deduction. In this way, a "calculus of natural deduction" was obtained.

Besides the BHK interpretation, early inferentialist notions of validity also incorporated elements from proof theory, particularly Gentzen's work. They are usually formulated in a natural deduction framework and draw inspiration for their notions of canonicity and harmony from normalization and similar results on the proof theory of natural deduction. Because of their connection with proof theory and their strong association with the BHK proof interpretation, these inferentialist notions of validity are currently investigated under the name "proof-theoretic semantics" (Schroeder-Heister 2018).

4. The origin of Gentzen's formulation of natural deduction from a historical and conceptual perspective was investigated by von Plato (2012, § 5). He explored the possibility that Gentzen was inspired, not from the BHK interpretation directly, but from the axiomatic systems of Hilbert and Bernays (1934) and Heyting (1930). According to him, these axiomatic systems were coupled with Gentzen's intuition that "actual mathematical reasoning proceeds by hypotheses or assumptions, rather than instances of axioms."

1.3 Meaning, rules and validity

Even in purely deductive contexts, validity is not *sufficient* for successful argumentation. There is no doubt, however, that validity is a *necessary* condition of correct reasoning and cogent argumentation. Therefore, the study of validity is significant, although not exhaustive, for the study of our argumentation practices.

Validity is a semantic concept. In order to show validity of an argument, the *meaning* of the logical constants must be taken into account. It is by appeal to the meanings of the logical expressions that we can argue for the correctness of an inference from assumptions to conclusion in a valid argument. Therefore, the theory of meaning has a fundamental impact on the corresponding concept of validity.

Currently, the prevalent meaning theory is *denotational*: it explains meaning on the basis of *reference* and *denotation*. As a result, most students of logic are acquainted with the concepts of interpretation, valuation, satisfaction, truth value (understood as the denotation of propositions) and others notions associated with model-theoretic semantics, a particular kind of denotational semantics that borrows much of its technical notions from model theory.

The development of model-theoretic semantics represented a notable change of attitude in logical investigations. This may be hard to see because many logic textbooks, following a common trend in the field, present model-theoretic semantics as a natural extension of the early developments of modern symbolic logic. But there was indeed a drastic change of approach to the explanation of logical validity. The distinction between syntax and semantics, for instance, was absent in the early days of modern symbolic logic. Thus, although Frege and Russell made heavy use of symbolic notation, the symbols in their symbolic languages were never intended to be dissociated from their meanings: the formal languages were presented as a *notation* for expressing logical notions and relations. They were very careful, when introducing logical notions and relations, to explain their meaning by means of examples and by describing their general behavior.

Hilbert (1922) was the first to take important steps towards the separation of syntax from semantics when he proposed, for the sake of pursuing his consistency proof, that the symbolic systems of Russell and Frege be viewed purely as syntactical systems. Since then, symbolic logic started to change from logical investigations made more precise with the *use* of symbols to investigations *about* the symbolic systems themselves. For some time, Carnap (1934, § 1) subscribed to an interesting variant of this kind of formalism:

> The common view is that syntax and logic, despite their connections, are fundamentally theories of very different kinds. [...] It has been established through its development in the last decades, however, that logic can be rigorously handled only when it does not relate to judgements (thoughts or thought contents), but instead to the linguistic expressions, particularly the sentences.

Carnap (1934) attempted to answer traditional logical problems by developing a theory of pure logical syntax. Later, Tarski (1936) introduced the notion of model[5] in order to overcome what he saw as shortcomings of the syntactic approach to logical consequence. After Tarski's work, model-theoretic semantics joined the (syntactic) proof theory of the Hilbert school and became an indispensable part in modern logical theories. Logic became twofold: syntax and semantics.

With model theory taking care of the semantics, the common conception is that the syntax ought to be understood as the pure combinatorics of symbols. The familiar recursive definition of the well-formed formulas, for example, is a known part of the syntax. But in modern logical theories, the syntax does not concern itself merely with the construction (or specification) of formal languages. The formal proofs of a deductive system, especially axiomatic deductive systems, are understood as transformations and operations on strings of symbols completely devoid of meaning and are thus also a part of syntax. However, just as the grammars of formal languages are based on the semantic role of their syntactic units[6], the formulation of the logical rules and axioms in formal systems is not arbitrary, but instead guided by reflections on the meaning of the logical expressions.

Most expositions of model-theoretic semantics encourage a formalist stance towards syntax and proceed as if it was independent from the semantics, which is why so much emphasis is put on results that bring them together, such as completeness and soundness. As a result, there is an implicit dichotomy between syntax and semantics and a general belief that syntax is somehow opposed to meaning. But is formalism the only approach to syntax? Are rules and the practice of following rules essentially opposed to meaning?

5. The notion of model as originally used by Tarski (1936) differs substantially from the notion of model currently used in model theory. There is no doubt, however, that Tarski's work was the most important inspiration.

6. For instance, in a formal grammar, the symbols of the formal language are classified into classes that indicate their general semantic role—individual constants are used to denote objects of the domain, monadic predicate letters denote subsets of the domain whose elements are objects that have a certain property and so on.

As remarked above, some important pioneers of modern symbolic logic did not assume a formalist stance towards their symbolic systems. Moreover, judging from some passages in his work, even Hilbert (1928), who some consider to be the father of formalism, viewed formal proofs as an expression, or representation, of meaningful thought:

> The game of formulas, one that Brouwer judges so harshly, has a general philosophical significance, besides its mathematical value. This formal game is devised according to certain rules that express the *technique of our thinking*.

He seems to have believed that the syntactic rules used to construct formal proofs are not mere symbolic manipulation but an expression of our deductive practices.

Besides encouraging a formalist approach to syntax, model-theoretic semantics also changed significantly the way validity is established. In ancient Greece, Aristotle (Prior Analytics, 29b) argued intuitively for the validity of some deductive relations and later established the validity of a group of other deductive forms by reducing them to the first ones. In contrast, model-theoretic semantics explains validity in terms of quantification over models and, consequently, all valid forms are, conceptually, on the same level.[7] The pervasiveness of model-theoretic semantics and its dichotomy between syntax and semantics has shifted the attention of the logician away from inferences and deductions and has placed it instead into valuations and models. This has gone to such an extent that one needs to write a handful of paragraphs just to explain how inferences, as steps in a deductive proof or argument, can ascribe meaning to logical expressions and how proof theory, as the study of such deductive practice, can be a base for semantics.

> Proof-theoretic semantics [...] uses ideas from proof theory as a mathematical discipline, similar to the way truth-condition semantics relies on model theory. However, just this is the basis of a fundamental misunderstanding of proof-theoretic semantics. To a great extent, the development of mathematical proof theory has been dominated by the formalist reading of Hilbert's program as dealing with formal proofs exclusively, in contradistinction to model theory as concerned with the (denotational) meaning of expressions. This dichotomy has entered many textbooks of logic in

7. For example, there is no conceptual difference between the validity of *modus ponens* and that of an argument form with an infinite class of assumptions: both are predicated on models of the assumptions being models of the conclusion.

> which semantics means model-theoretic semantics and proof theory denotes the proof theory of formal systems. The result is that proof-theoretic semantics sounds like a contradiction in terms even today. (Schroeder-Heister 2006, § 1)

I do not use the term "proof-theoretic semantics" to refer to a semantics of proof, or to a semantics where truth is replaced by proof. Instead, I use it to designate a family of semantics for logic that, although indeed inpired from results in proof theory, are based on the idea that meaning should be explained not in terms of *denotation* but in terms of *use*. To explain meaning in terms of use in the context of a semantics for logic is to adopt the view that certain deductive rules implicit in our linguistic practice determine the meaning of the logical constants. Thus, in my view, proof-theoretic semantics is to a meaning theory based on use what model-theoretic semantics is to a meaning theory based on denotation.

Perhaps it would be useful to illustrate the proof-theoretic approach to meaning, as examined in detail in following chapters, by means of an example with a single logical constant: implication. Since I am concerned with a semantics for logic, the relevant practice is *deductive* practice; the relevant use is *deductive* use. In contrast with strictly formalist views commonly associated with axiomatic systems, where purely formal concerns like economy of axioms and rules are prominent, natural deduction enables a more reflective strategy to capture the deductive behaviour of the logical constants. There are two aspects to the use of implications in natural deductions: they can appear as conclusion or as premisses of inferences.

There are many ways in which implications can appear as either premiss or conclusion of inferences, but not all of them are essential to the meaning of implication. There are essential, or canonical, uses of implication either as premiss or as conclusion of an inference, and there are also inessential, derivative uses. In the context of conclusions, we can express the canonical uses by especifying the necessary and sufficient conditions for inferring a sentence with implication as the main logical connective.

The introduction rule for implication in natural deduction can be seen as expressing exactly these necessary and sufficient conditions. Thus, a necessary and sufficient condition for making an inference whose conclusion is $A \rightarrow B$, where A and B are arbitrary sentences, is that there is a deduction of B from assumption A. Of course, there are other situations in which $A \rightarrow B$ may appear as conclusion of an inference. But these other uses are inessential and can be explained by reference to the canonical use (section 2.3).

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}$$

Similarly, the elimination rule for implication can be seem as the canonical way to infer consequences from sentences with implication as their main logical connective. The elimination rule expresses what consequences must be accepted on the strength of $A \to B$ and the auxilliary premiss A. Again, there are other consequences that can be extracted from $A \to B$ besides those of the corresponding elimination rule. They are inessential.

$$\dfrac{A \quad A \to B}{B}$$

By carrying the considerations just sketched to the other logical constants, we can show that any valid deductions can be accounted for by reference only to canonical inferences. In other words, any deductive relatioship between sentences can be established using the introduction and eliminations rules of the logical constants involved. Furthermore, there is a striking relationship between both aspects of the use of implication: what was required for the introduction of $A \to B$, namely, a derivarion of B (based on assumption A) can be restored by applying the elimination rule. To put it in another way, what is obtained by elimination of $A \to B$ was already at hand if we assume $A \to B$ to have been derived by the introduction rule. The relationship between the deductive behavior of introduction and elimination rules for a logical constant can be studied in order to uncover important semantic properties. A general study of this kind constitutes the core of the proof-theoretic approach to the semantics of logic.

Traditionally, however, proof-theoretic semantics has often been called to fulfil also other objectives and to answer to other intuitions. Verificationism in the theory of meaning and intuitionism in the philosophy of mathematics are the most notable among them. In contrast, this text was written under the background conviction that these matters are completely orthogonal to the issue of validity and the meaning of the logical constants. In particular, I am not sure whether the conflict in the philosophy of mathematics does not involve strictly *mathematical* (non-logical) considerations but instead solely revolves around what is the correct underlying logic, as Dummett believed.[8] In

8. Philosophical views associated with mathematical constructivism can be very diverse

addition, I think that an analysis of the concept of proof, which is admittedly an important part, is not sufficient to account for deductive practice (in mathematics and other areas of discourse). The doctrine of verificationism, with its focus on assertion and assertability, is also too narrow to account for the richness of our deductive practices. More precisely, the following theses are not endorsed:

Assertability The thesis that assertions are the central linguistic concept of a meaning theory based on use and, consequently, the view that inferences are transitions from assertions to assertions. This is often expressed in the slogan that a meaning theory based on use substitutes, in the general framework of a denotational meaning theory, the concept of truth conditions for assertability conditions.

BHK interpretation The thesis that the BHK interpretation of the logical constants should be considered the starting point for a complete and coherent semantic explanation of the meaning of the logical constants and, in particular, the view that the BHK interpretation corresponds somehow to the introduction rules of natural deduction.

My primary motivation in the next chapters is to pursue a proof-theoretic explanation of validity which is still faithful to the idea of "meaning as use" but does not subscribe to the theses above.

and rich. They sometimes involve positions with regard to some strictly mathematical concepts. Troelstra and van Dalen (1988, § 1.4) offer a concise survey of the most important philosophical positions associated with constructivism in mathematics.

2 The Deductive Use of the Logical Constants

As already remarked, the syncategorematicity and universality of logic makes it particularly suitable as a test case to evaluate the tenability of a systematic approach to the idea of "meaning as use". Deductive argumentation patterns can be more easily formulated in terms of rules expressing the deductive use of the logical constants than the usage patterns of most other natural language expressions. Inevitably, these rules would not capture every subtle aspect of the use of the natural language expressions corresponding to the logical constants. How tight must usage patterns and rules match is a question of balance between theoretical idealisation and pragmatic robustness.

This chapter examines the meaning of the logical constants as expressed by the inference rules of natural deduction and briefly assess these rules in light of our deductive practices. The discussion is confined to the propositional logical constants (no quantifiers). The deductive harmony between introduction and eliminations rules is also addressed, since this is a fundamental feature behind proof-theoretic definitions of validity.

2.1 Preliminaries

This section can be skipped and subsequently consulted on demand for clarifications around notation and terminology.

The language. I cover propositional languages with infinitely many propositional variables (atomic sentences) and the propositional logical constants: $\to$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction) and $\bot$ (absurdity). The complex sentences of the language are formed from atomic sentences by means of composition with the logical constants in the usual way. Latin letters (A, B, C etc.) are used to stand for arbitrary sentences of the language (small letters indicate atomic sentences) and capital Greek letters (Γ and Δ) to stand for finite collections of sentences.[1] Subscripts are used whenever it is necessary or

1. I use the more ambiguous term "collection" because I want to leave open whether sentences are collected into sets or, perhaps, multisets (section 8.3). No discussion or result

convenient. The *degree of a sentence* is the number of logical constants that occur in it.

Arguments Formally, *arguments* can be considered as trees of sentence occurrences (designated with Π, possibly with subscripts). They are constructed from top to bottom, from the leaves to the root, by *inferences*. These inferences lead from one or more sentences, the *premisses*, to a single sentence, the *consequence* (usually separated by an inference line). In an argument, each premiss of an inference is either a leaf of the tree or the consequence of a previous inference. Thus, argument trees are formal representations of the process of argumentation, or reasoning, with some leaves acting as *assumptions* and the root acting as the *conclusion* of the argument. Any occurrence of a sentence in an argument determine, in the obvious way, a subargument with that sentence as conclusion.[2] A *path* in an argument is a sequence of sentence occurrences such that each sentence in the path is an immediate inferential consequence of the previous one.[3] Every leaf in an argument is initially an assumption, albeit assumptions can be discharged by inferences.[4] After an assumption is discharged by an inference, the argument, starting from the consequence of that inference, does not depend any more on the assumption. The discharge of assumptions are indicated using square brackets with numeric indices used to pinpoint the particular inference discharging the assumption. Whenever it is clear from context, the numeric indices are left implicit. As a convention, sentences occurring, without an inference line, immediately above (below) Π indicate leaves (the root) of Π. I write $\langle \Gamma, A \rangle$ to denote an argument from assumptions Γ (those that remained undischarged throughout the argument) to conclusion A without paying attention to the argumentation process that goes from Γ to A. Also, for the sake of simplicity, I often talk about sentences when I actually mean occurrences of sentences in an argument, and similarly with respect to inference rules and the particular inferences resulting from their application.

Derivations Propositional intuitionistic logic is characterised by the standard system of natural deduction (Gentzen 1934; Prawitz 1965). The inference

is affected by the choice.

2. As a limiting case, a single sentence occurrence is an argument with that sentence acting as both assumption and conclusion.

3. Again, as a limiting case, a single sentence occurrence measures an *empty* path from that sentence occurrence to itself.

4. An axiom or logical theorem A can be considered the result of an inference from leaf A to conclusion A that discards the leaf occurrence of A.

rules for the propositional connectives are symmetrically distributed between introduction (I) and elimination (E) rules.

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\ {\to}\mathrm{I} \qquad\qquad \dfrac{A \quad A \to B}{B}\ {\to}\mathrm{E}$$

$$\dfrac{A \quad B}{A \wedge B}\ \wedge\mathrm{I} \qquad\qquad \dfrac{A \wedge B}{A}\ \wedge\mathrm{E} \qquad \dfrac{A \wedge B}{B}\ \wedge\mathrm{E}$$

$$\dfrac{A}{A \vee B}\ \vee\mathrm{I} \qquad \dfrac{B}{A \vee B}\ \vee\mathrm{I} \qquad \dfrac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\ \vee\mathrm{E}$$

Negation ($\neg$) can be defined as usual in terms of implication and absurdity. The rule for the absurdity logical constant $\bot$ can be considered an elimination rule.

$$\dfrac{\bot}{A}\ \bot\mathrm{E}$$

Natural deduction derivations are a particular subclass of arguments in which every inference is in accordance with one of the inference rules above.

Systems A collection of inference rules determine a deductive system S. The main deductive system N is the natural deduction system for propositional intuitionistic logic. A subscripted list of constants indicate fragments thereof (for example, $\mathsf{N}_{\to}$ for the implication fragment). These deductive systems can be supplemented with a *basic system* B: a collection of rules for the derivation of atomic sentences from atomic sentences. A basic rule b has the form

$$\dfrac{a_1 \cdots a_n}{b}$$

where $a_1 \cdots a_n$ and b are atomic sentences. Except for some odd remarks (exceptions always explicitly stated), all the bases considered are *production systems*, i. e. its basic rules do not discharge assumptions.[5]

5. Exception is made to basic axioms, since my interpretation of axioms involve discharge (footnote 4).

2.2 The theory and practice of deduction

A natural deduction system consists of a collection of inference rules designed to capture the concept of deduction. The most interesting feature of natural deduction compared to other deductive systems is the classification of its inference rules between introduction rules and elimination rules (at least one of each kind for each one of the logical constants). Natural deduction rules for a certain logical constant always figure a sentence with that constant as the main operator and also its subsentences. Moreover, as a general pattern, the subsentences occur as premisses in the introduction rules whereas in the elimination rules it is usually the other way around.[6]

The complementary pattern between introductions and eliminations naturally gives rise to the interpretation that the introduction rules for a logical constant γ express the necessary and sufficient conditions under which we can infer a sentence containing γ as the main logical connective. Analogously, the elimination rule for γ express what are the consequences that can be extracted from a sentence containing γ as the main logical connective, together with other, minor, premisses when necessary. This feature lends plausibility to the idea of introduction and elimination rules as *semantic* explanations, something that could hardly be claimed of some arbitrary syntactic system. That is not to say, however, that the *standard* introduction and elimination rules for the logical constants are entirely in agreement with practice.

2.2.1 Implication

This connective is perhaps the most complicated and controversial of all the logical constants. In an implication $A \to B$, the component sentences are asymmetrically connected, with A the *antecedent* and B the *consequent*, in contrast with other connectives like conjunction where it makes no difference which conjunct occurs first. Its most common English reading is "if A then B". In general, when we use this expression, we claim a certain relation of entailment between antecedent and consequent: one follows from the other by causality, deduction or other kind of chain of plausible reasons. However, the meaning attached to implication by the introduction rule is somewhat weaker

6. In fact, the situation with the elimination rules is a little more complicated. The sentence containing the logical constant as main operator occur as a *major premiss* with subsentences sometimes also figuring as *minor premisses*. A subsentence may (as in inference rules) or may not (as in deduction rules) occur as conclusion of the rule (Prawitz 1965, § I.2.B).

than the meaning usually associated with the "if A then B" expression.

$$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B} \to\mathrm{I} \qquad\qquad \frac{A \quad A \to B}{B} \to\mathrm{E}$$

As a matter of fact, the introduction rule for implication *does not require* that an assumption of the form A be actually discharged. Thus, the rule permits the inference of $A \to B$ on conditions under which A is irrelevant. On the other hand, the elimination rule for implication *does require* a premiss of the form A for its application. From the point of view of the introduction rule, the stronger requirement for elimination is natural since, even assuming the major premiss to have been obtained by $\to$I, we cannot tell in advance if an assumption was discharged by its application. However, from the point of view of the elimination rule, a stronger meaning can be assigned to implication since A is always available but would go unused in a deduction of B not depending on A.

The unbalance between the introduction and elimination rules for implication and the disagreement between those rules and the expressions said to be its equivalent in natural languages has led some authors, notably Tennant (1987, chapter 17), to favour a relevant reading of implication.

2.2.2 Conjunction

Especially in contrast with implication, the rules for conjunction are perhaps the most straightforward and uncontroversial. This is no surprise, since the meaning that the introduction and elimination rules assign to conjunction is very narrow.

$$\frac{A \quad B}{A \wedge B} \wedge\mathrm{I} \qquad\qquad \frac{A \wedge B}{A} \wedge\mathrm{E} \qquad\qquad \frac{A \wedge B}{B} \wedge\mathrm{E}$$

A simple, although not entirely exact, intuitive explanation of the rules is to say that a conjunction allows us to convey, in a *single* sentence of the form $A \wedge B$, the exact same information conveyed by both sentences, A and B. This narrow meaning is seldom, if ever, intended in ordinary speech where expressions like "and" and "but" are more often used to convey more information than mere logical conjunction: temporal sequence, surprise, disbelief and so on.

2.2.3 Disjunction

In English, disjunction is usually associated with the meaning of the expression "or". In ordinary conversations, we often use "or" to express an exclusive (or, sometimes, inclusive) choice or option as in "you must sing the tune either in D major or F major, otherwise it is not good" or "the bank allows you to transfer money in person or via the internet". Thus, it is common to use "or" in contexts involving agents and actions. Yet, disjunction also appears in more declarative contexts, especially in situations when we do not have enough information to determine which one of the disjuncts holds. Even in these situations, we can still extract consequences from the disjunction by showing them to be derivable from each one of the disjuncts, as is expressed by the elimination rule.

$$\frac{A}{A \vee B}\ \vee\mathrm{I} \qquad \frac{B}{A \vee B}\ \vee\mathrm{I} \qquad \frac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}\ \vee\mathrm{E}$$

On the other hand, at least from an epistemological point of view, the introduction rule for disjunction is somewhat pointless. The conclusion $A \vee B$ does not say which one of the disjuncts holds. So, epistemologically, there seems to be more information on the premiss of the rule than on its consequence. Arguably, this can also be said of $\wedge$E. But, given that $\wedge$E is an elimination rule and its purpose is to extract consequences from its premiss, it is expected that those consequences may have less information.

2.2.4 Negation

In most of the modern texts that employ natural deduction, negation is a defined symbol. It is defined in terms of the constant $\bot$, usually called "absurdity" or "falsum", and implication: $\neg A \equiv A \to \bot$. A noteworthy exception is Gentzen (1934, § 2.21), who gives introduction and elimination rules directly for negation. I follow the current practice and give rules for the absurdity constant, letting negation stand defined.

$$\frac{\bot}{A}\ \bot\mathrm{E}$$

Negation is a very interesting and controversial logical constant. Its classical understanding goes back to Aristotle (On Interpretation, VI) who believed that every meaningful sentence had a meaningful contradictory negation which could, in principle, be asserted.

Logical negations are notably very narrow in comparison with natural language negative expressions like “not” or “no” in English. Negative expressions in natural languages can be used for a very wide range of functions, most of which do not correspond entirely with a single logical conception of negation (Horn and Wansing 2017, § 1). For instance, “not” can be used to express contrariness instead of contradictoriness. It can also be used to deny, a speech act arguably distinct from asserting the contradictory.

Over the years, the richness of negative expressions in natural languages drove the development of a plurality of logical systems. Those that purport to capture contradictory negation mostly adhere to the core principle of non-contradiction. This principle is often itself understood in negative form, as a prohibition of maintaining both A and $\neg A$ at the same time. With negation defined, ⊥E could be interpreted as punishment for violating the prohibition against contradiction: by →E and ⊥E, arbitrary sentences become deducible and we have deductive triviality. In actual argumentative contexts, however, the absurdity constant would preferably never be eliminated (staying only as a kind of threat). Usually, when facing contradictions, after obtaining absurdity as consequence, →I would be applied in order to provide a refutation of a hypothesis (under the remaining assumptions).

A troublesome issue with this account of negation is that ⊥ does not follow the same pattern of introduction and elimination rules as the other logical constants. In order to remedy the situation, Dummett (1991, chapter 13) has proposed the following introduction rule for ⊥, where a_i ranges through all atomic sentences of the language:

$$\frac{a_1 \quad a_2 \quad a_3 \quad \cdots}{\bot}\ \bot\mathrm{I}$$

With ⊥E considered as an elimination rule, Dummett believes that ⊥I above is the more harmonious (section 2.3) introduction rule. If we assume, without loss of generality, the consequence of ⊥E to be atomic, an application of the rule would allow us to obtain *any* atomic sentence. Therefore, it is plausible that an introduction rule in harmony with ⊥E should require no less than all atomic sentences as premisses. Still, given the potentially infinite number of premisses, it is hard to maintain that ⊥I is, in any intuitive sense, a legitimate inference rule that could be employed in actual deductive practice.

2.3 Deductive Harmony

As already remarked, the introduction rules for a constant γ can be viewed as an expression of the necessary and sufficient conditions for inferring its conclusion $A\,\gamma\,B$ (assuming the paradigmatic case where γ is a binary connective). Now, there is a very plausible requirement that can placed on the corresponding elimination rules: the consequences extracted from its major premiss $A\,\gamma\,B$ can never extrapolate what was necessary for its inference by means of the introduction rules. A similar requirement can be placed on the introduction rules from the point of view of the elimination rules: given the context, whatever can be deduced from the conclusion by means of the elimination rule could already be deduced from the premisses. When both these requirements are fulfilled, the introduction and elimination rules for a logical constant are in *harmony* with each other.

The concept of harmony between logical rules goes back to a much quoted passage from Gentzen (1934, II. § 5.13) to the effect that "the introduction rules are definitions and the eliminations are only their consequences thereof". Adopting ideas and terminology from Lorenzen (1955, § 4), Prawitz (1965, chapter II) attempted to make Gentzen's remarks more precise by formulating an *inversion principle*, the cornerstone for his normalisation procedures for natural deduction systems.

Normalization procedures rely on *reductions*. They enable the removal of roundabouts in a natural deduction derivation. More precisely, in a derivation Π of G from Γ, when an application of an introduction rule have as consequence the major premiss of an application of an elimination rule, there is a reduction which results in a derivation Π' of G from, at most, the same assumptions Γ without going through those steps. There is a reduction for each pair of introductions and eliminations.

($\to$)

$$\begin{matrix}\dfrac{\begin{matrix}\Pi_1\\ A\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \Pi_2\\ B\end{matrix}}{A \to B}}{B}\\ \Pi_3\end{matrix} \qquad\qquad\qquad \begin{matrix}\Pi_1\\ A\\ \Pi_2\\ B\\ \Pi_3\end{matrix}$$

($\wedge$)

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\Pi_1\\ A\end{matrix} \quad \begin{matrix}\Pi_2\\ B\end{matrix}}{A \wedge B}}{A}\\ \Pi_3\end{matrix} \qquad\qquad\qquad \begin{matrix}\Pi_1\\ A\\ \Pi_3\end{matrix}$$

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{cc}\Pi_1 & \Pi_2\\ A & B\end{array}}{A \wedge B}}{B} \\ \Pi_3 \end{array} \qquad\qquad \begin{array}{c}\Pi_2\\ B\\ \Pi_3\end{array}$$

(∨)

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A\end{array}}{A\vee B} \quad \begin{array}{c}[A]\\ \Pi_2\\ C\end{array} \quad \begin{array}{c}[B]\\ \Pi_3\\ C\end{array}}{C}\\ \Pi_4\end{array} \qquad\qquad \begin{array}{c}\Pi_1\\ A\\ \Pi_2\\ C\\ \Pi_4\end{array}$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ B\end{array}}{A\vee B} \quad \begin{array}{c}[A]\\ \Pi_2\\ C\end{array} \quad \begin{array}{c}[B]\\ \Pi_3\\ C\end{array}}{C}\\ \Pi_4\end{array} \qquad\qquad \begin{array}{c}\Pi_1\\ B\\ \Pi_3\\ C\\ \Pi_4\end{array}$$

On the left, there is a derivation containing a roundabout: a logical constant is introduced just to be, immediately after, eliminated. Since the elimination rules are in harmony with the introduction rules, their application just restored what was already required of the premisses for the respective introduction rule. As a result, both inferences can be avoided by rearranging the derivation as shown on the right.

Under certain conditions, harmony between the introduction and elimination rules for a logical constant guarantees that its addition to a deductive system yields a *conservative extension*. In conservative extensions, the addition of new expressions to the language—in this context, logical constants with their pairs of introduction and elimination rules—leaves the meaning of the other expression unharmed. In particular, the extended system does not enable new derivations where the added expression does not occur that were not already available before: the new expressions extended the system conservatively without revising the meaning of old expressions. Now, harmonious inference rules would generally provide for conservative extensions. For, suppose the conditions for the application of the introduction rule for the newly added constant were fulfilled. Then, if harmony obtains, the elimination rule does not enable the derivation of new consequences besides those that were already derivable in the original system.

The fact that the addition of the inference rules to a deductive system, or, for that matter, to any comprehensive and coherent linguistic practice, yields a conservative extension provides evidence for the *logicality* of these rules. Otherwise, if the addition of rules for the use of a logical constant γ change the original system in such a way that a sentence A, not containing γ, now becomes derivable, then there is strong evidence that γ incorporates some extralogical content. Consider, for instance, an original system composed solely of descriptive expressions. In such a case, the rules for γ could license the derivation of a descriptive (since it does not contain γ) sentence that was not previously derivable.

Moreover, seen as a requirement, harmony can avert problematic connectives like the infamous `tonk` (Prior 1960). In contemporary discussions, `tonk` is usually specified as a connective with the introduction rules of disjunction and the elimination rules of conjunction.

$$\frac{A}{A\texttt{tonk}B}\ \texttt{tonk}\mathrm{I} \qquad \frac{B}{A\texttt{tonk}B}\ \texttt{tonk}\mathrm{I} \qquad \frac{A\texttt{tonk}B}{A}\ \texttt{tonk}\mathrm{E}\ \frac{A\texttt{tonk}B}{B}\ \texttt{tonk}\mathrm{E}$$

The possibility of defining `tonk` through inference rules is sometimes weighted against inferentialist approaches to the meaning of logical constants. Naturally, just as with approaches based on truth (think, for instance, of bivalence), background assumptions must come into play (Belnap 1962). Thus, it is easy to show that admission of `tonk` into the language trivialises the deductive practice. But it is also easy to show that the inference rules for `tonk` are not harmonious: the elimination rules for `tonk` enable the derivation of more than what is required to obtain their major premiss by an introduction rule.

The requirement of harmony protects inferentialism against overpermissive introduction and elimination rules. A protection against overstringent ones (what is sometimes called "weak disharmony") can also be altogether desirable. The dual connective `knot`, with the introduction rules for conjunction and the elimination rules for disjunction, can be used to illustrate this predicament (Došen and Schroeder-Heister 1985, § 2).

$$\frac{A \quad B}{A\texttt{knot}B} \qquad\qquad \frac{A\texttt{knot}B \quad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C}$$

Although the elimination rule for `knot` enables inferences *only* of sentences already required by the introduction rule, it does not enable inferences of *all* of those sentences: the elimination rule for `knot` is more stringent than it needs to be. The further requirement that introduction and elimination rules must explore their full potential (of course, under harmony restrictions) is often called *stability* (Dummett 1991, chapter 13). It imposes a stricter balance between the introduction and elimination rules. Intuitively, harmony ensures soundness while stability ensures completeness.

As already remarked, the reductions employed in normalization procedures are based on the idea that there is nothing to be gained by introducing and then, immediately after, eliminating a sentence. Their existence show that the elimination rules do not extrapolate the necessary conditions for the introduction rules. Now, the reverse side of reductions are *expansions*.

($\rightarrow$)

$$\begin{array}{c}\Pi_1\\ A \rightarrow B\\ \Pi_2\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A \rightarrow B\end{array}\quad [A]}{B}}{A \rightarrow B}\\ \Pi_2\end{array}$$

($\wedge$)

$$\begin{array}{c}\Pi_1\\ A \wedge B\\ \Pi_2\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A \wedge B\end{array}}{A}\quad \dfrac{\begin{array}{c}\Pi_1\\ A \wedge B\end{array}}{B}}{A \wedge B}\\ \Pi_2\end{array}$$

($\vee$)

$$\begin{array}{c}\Pi_1\\ A \vee B\\ \Pi_2\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ A \vee B\end{array}\quad \dfrac{[A]}{A \vee B}\quad \dfrac{[B]}{A \vee B}}{A \vee B}\\ \Pi_2\end{array}$$

The expansions, on the other hand, show that the elimination rules supply the sufficient conditions for regaining the major premiss through the introduction rules. They ensure that it is possible to eliminate a sentence and subsequently introduce it without any loss. Consequently, unstable connectives like `knot` are unable to produce satisfactory expansions.

$$\begin{array}{c}\dfrac{\begin{array}{c}\Pi_1\\ A\texttt{knot}B\end{array}\quad \dfrac{[A]\quad B}{A\texttt{knot}B}\quad \dfrac{A\quad [B]}{A\texttt{knot}B}}{A\texttt{knot}B}\\ \Pi_2\end{array}$$

For instance, notice that the purported expansion above introduces further assumptions A and B which may not be assumptions of Π_1.

The notions of harmony and stability are often informally discussed. Their underlying idea of "no more, no less" is thus open to interpretation. Moreover, even inference rules usually considered to be harmonious and stable can arguably display some unbalance, for example, the mismatch mentioned in section 2.2.1 between the introduction and elimination rules for implication. Inferentialist definitions of validity offer an excellent opportunity to cash out the notions of harmony and stability in a more precise manner.

3 Early Notions of Proof-Theoretic Validity

The BHK interpretation does not lend itself easily to the recursive treatment required for a definition of validity. The clause for implication, for instance, seems to be strongly impredicative. It refers to *any* construction of the antecedent, and that might involve a construction of the implication itself, for example, in a roundabout proof (Dummett 2000, § 7.2).

Nevertheless, relying on a notion of *canonicity* made viable by the deductive harmony of the rules in natural deduction, Prawitz (1971, 1973, 1974, 2006) proposed proof-theoretic inductive definitions of validity intended for justifying predicate intuitionistic logic. His definitions were advanced from a constructive standpoint, with the BHK interpretation regarded as the authoritative explanation of meaning of the logical constants. The following discussion of the evolution of these notions of validity is limited to the propositional case.

3.1 Atomic bases and monotonic extensions

The BHK clause for implication refers to proof constructions of the antecedent and consequent. A fully recursive clause, going all the way down to atomic components, seems to require a base system to provide proof constructions for atomic sentences. In his first proposal, Prawitz (1971, § IV.1.1) considered atomic production systems (or Post systems, as he called them) for the task. As it turned out, relying on atomic bases for the proof constructions of atomic sentences required some adjustments (quote superficially altered for notation):

> I shall thus speak of a construction k of a sentence A relative or *over* a Post system S. When A is atomic such a construction k will simply be a derivation of A in S. [...] when relativised to S, a construction k of $A \to B$ over S where A and B are atomic will be a constructive (or with Church's thesis: recursive) function that transforms every derivation of A in S to a derivation of B in S. However, a consequence of such a definition would be that if A is not constructible over S (i. e. not derivable in S), $A \to B$ is

> automatically constructible over S since any constructive function would vacuously satisfy the condition [...]. In particular, provided $\bot$ is not constructible over S and $\neg A$ is a shorthand for $A \to \bot$ as usual, it follows that there is no system S over which $\neg\neg A \to A$ is not constructible (hence, classically, $\neg\neg A \to A$ is constructible in every S), which is clearly contrary to the constructive interpretation of implication and negation.

Prawitz's adjustment was to consider extensions S' of the base system S. As a result, antecedents occasionally not constructible over S become constructible over some extension S' of S. The intuitive interpretation behind this approach to atomic systems is to regard them as evolving knowledge bases. The paradigmatic model is mathematical knowledge. Consequently, the extensions are required to be *monotonic*: any atomic sentence constructible over S stays constructible over extensions S'. This corresponds to the intuition that a theorem, once proved, remains proved.

3.2 Validity for derivations

There were many attempts to put intuitionism into a constructively acceptable formal footing. They range from realizability (Kleene 1945), developed in the context of the study of general recursive functions, to attempts directly inspired by the BHK interpretation, like the theory of constructions (Kreisel 1962).

Among the difficulties encountered by formal approaches based on BHK are the interpretation of the notion of construction, the specification of constructions for the atomic sentences and the potential impredicative characther of the clause for implication. Early proof-theoretic notions of validity attempted to answer these challenges with reduction procedures, atomic bases and normalisation results. The atomic basis B provide the constructions for the atomic sentences, the constructive transformations are provided by reduction procedures and the charge of impredicativity is neutralised by normalisation (slightly adapted from the source):

Definition 3.2.1 (Prawitz 1971, § A.1.2)**.** A closed derivation Π for conclusion A is *valid in* B if, and only if

- A is atomic and Π reduces to a derivation in B; or
- A is of the form $B \wedge C$ and Π reduces to a derivation of the form

$$\frac{\Pi_1 \quad \Pi_2}{B \wedge C}$$

where Π_1 is a derivation of B and Π_2 is a derivation of C and they are valid in B; or

– A is of the form $B \vee C$ and Π reduces to a derivation of the form

$$\frac{\Pi_*}{B \vee C}$$

where Π_* is a derivation of B or C and it is valid in B; or

– A is of the form $B \to C$ and Π reduces to a derivation of the form

$$\frac{\begin{matrix}\Pi_2 \\ B\end{matrix}}{A \to B}$$

such that for each extension B' of B and for each closed derivation

$$\begin{matrix}\Pi_1 \\ A\end{matrix}$$

that is valid in B', it holds that

$$\begin{matrix}\Pi_1 \\ [A] \\ \Pi_2\end{matrix}$$

is a derivation of B and it is valid in B'.

Definition 3.2.2 (Prawitz 1971, § A.1.2)**.** A derivation Π is *valid in* B if, for each extension B' of B, where Π' is the result of replacing every assumption A in Π by a valid closed derivation in B', Π' is valid in B'.

A notion of validity is thus defined relative to an atomic basis B. Logical validity is then obtained, through generalisation, as validity in every atomic basis.

3.3 From derivations to arguments

Roughly, the problem with definitions 3.2.1 and 3.2.2 is that they apply solely to derivations, but not to arbitrary arguments (section 2.1). A genuine semantic notion of validity, on the other hand, must account for arbitrary arguments, whereby valid arguments can unfold through whatever valid means are available and not only by the neat introductions and eliminations of natural deduction derivations. In other words, a notion of validity must determine whether a conclusion can be logically deduced, or follows logically, from certain assumptions (or, even, perhaps, from no assumptions) irrespective of the form of inferences that lead from assumptions to conclusion.

Nevertheless, for arguments that are already derivations, the definitions indeed work as expected: validity emerges as a consequence of the definitions and not from whatever status any inferences in the argument may have as applications of introduction or elimination rules. For example, consider the argument below.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (B \land C) \quad [A]}{B \land C}}{B}}{A \to B} \qquad \dfrac{\dfrac{\dfrac{A \to (B \land C) \quad [A]}{B \land C}}{C}}{A \to C}}{(A \to B) \land (A \to C)}$$

By definition 3.2.2, it is valid in a basis B if, for any extension B', the closed derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A \to (B \land C)\end{array} \quad [A]}{B \land C}}{B}}{A \to B} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_2\\ A \to (B \land C)\end{array} \quad [A]}{B \land C}}{C}}{A \to C}}{(A \to B) \land (A \to C)}$$

is valid in B', where Π_1 and Π_2 are closed derivations valid in B'. By definition 3.2.1, the subderivations for $A \to B$ and $A \to C$ must be valid in B'. Therefore for any closed derivations Π_3 and Π_4 of A valid in an extension B'', the subderivation for B and C are valid in B''. Now, provided that Π_1 and Π_2 are indeed valid in B', that is, that they reduce to the forms below, the subderivations for B and C do indeed reduce to valid derivations in B' by means of the reductions in section 2.3.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots \ \text{valid in } \mathsf{B}'\\ B\end{array} \quad \begin{array}{c}[A]\\ \vdots \ \text{valid in } \mathsf{B}'\\ C\end{array}}{B \land C}}{A \to (B \land C)}}{B \land C} \qquad \begin{array}{c}\Pi_3\\ A\end{array}}{B} \qquad \text{reduces to} \qquad \begin{array}{c}[A]\\ \vdots \ \text{valid in } \mathsf{B}'\\ B\end{array}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\ \text{valid in } \mathsf{B}'\\ B\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\ \text{valid in } \mathsf{B}'\\ C\end{matrix}}{B \wedge C}}{A \to (B \wedge C)}}{B \wedge C} \qquad \begin{matrix}\Pi_4\\ A\end{matrix}}{C} \qquad \text{reduces to} \qquad \begin{matrix}[A]\\ \vdots\ \text{valid in } \mathsf{B}'\\ C\end{matrix}$$

The closed derivations Π_3 and Π_4 valid in B'', but perhaps not valid in B', are discarded. This shows that the example is valid in an arbitrary basis B and is, therefore, logically valid.

Although the validity of the introduction rules are, in a sense, presumed in the clauses of definition 3.2.1, notice that the notion of validity affords, at the very least, a justification for the elimination rules. However, through the restriction to derivations, the elimination rules are, in an important sense, taken for granted. Furthermore, a notion of validity is supposed to sort out invalid arguments from valid arguments, but the restriction to derivations exclude potentially invalid arguments already at the outset.

In order to counteract these objections, Prawitz (1973, § 2) expanded his definitions to cover arbitrary arguments. Now, the difficulty is that the standard reductions do not work for arbitrary arguments but only for derivations.

3.4 From reductions to justifications

Even very simple arguments, for instance,

$$\frac{A \wedge (B \wedge C)}{A \wedge C}$$

are not properly handled, because, after replacing derivations for the assumption $A \wedge (B \wedge C)$, the result cannot be reduced to a derivation ending with an introduction as required by the respective clause of definition 3.2.1 (the valid inference from $A \wedge (B \wedge C)$ to $A \wedge C$ at the end cannot be unfolded through the standard reductions). In general, the same holds for arbitrary arguments containing valid inferences not spelled out in terms of introductions and eliminations (Schroeder-Heister 2006, § 5).

The solution proposed by Prawitz (1973, § 4) is to evaluate arguments coupled with *justifications*. In this context, the concept of justification is a generalisation of the standard reductions meant to apply more extensively to arbitrary arguments instead of being restricted to derivations. The justifications can then account for the intuition that, although no standard reduction is applicable, closed arguments for $A \wedge (B \wedge C)$ in the example above already contain, in a sense, valid arguments for $A \wedge C$ as required by the respective clause in definition 3.2.1.

For example, justifications can be considered to include uniform substitutions and compositions, beside reductions and expansions. Consequently, the validation of many arbitrary arguments that are not derivations can now be achieved.

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ A\end{array} \quad \dfrac{\begin{array}{c}\Pi_2\\ B\end{array} \quad \begin{array}{c}\Pi_3\\ C\end{array}}{B \wedge C}}{A \wedge (B \wedge C)}}{A \wedge C} \quad \text{justifiably transforms into} \quad \dfrac{\begin{array}{c}\Pi_1\\ A\end{array} \quad \begin{array}{c}\Pi_3\\ C\end{array}}{A \wedge C}$$

There can be also particular justifications associated with basic derivations. As the basis is extended, the justifications may have to be extended accordingly. For instance, particular transformations may be considered for the intuitionistic theory of iterated inductive definitions (Martin-Löf 1971, § 4).

Consequently, the definition of validity for arbitrary arguments must be formulated relative to bases and justifications. However, when the concern is logical validity, instead of particular mathematical or empirical applications, the justifications can mostly be left implicit. They would naturally include any constructively acceptable manipulation of arguments. For the exclusively logical material, general uniform substitutions and compositions probably suffice.

4 A Critique of Early Proof-Theoretic Notions of Validity

Prawitz proposed many formulations for his proof-theoretic definitions of validity. They were mostly minor variations and improvements on the definitions discussed in chapter 3. Prawitz's proposals became the standard reference for proof-theoretic semantics, and the literature on the subject usually gravitates around them. Although I do think of these proposals as standard in a historical sense, I believe they are unsatisfactory in many respects. My objections are both conceptual and technical. They are advanced from the standpoint of proof-theoretic validity as a notion of validity faithful to the idea of "meaning as use". While this standpoint is undoubtedly an important element of the standard proposals, it may well not be the only or even the prevalent one.

4.1 The placeholder view of assumptions

Schroeder-Heister (2008, § 3) pointed out some dogmas of proof-theoretic semantics. One of the dogmas was the *the primacy of the categorical over the hypothetical*, or, as it was latter called, the *placeholder view of assumptions*. According to this dogma, hypothetical arguments, or arguments from assumptions, should be reduced to closed arguments, or closed proofs (proofs from no assumptions). In other words, assumptions are considered to be placeholders for closed proofs. The proof-theoretic definitions of validity for arguments proposed by Prawitz (1971, 1973, 1974, 2006) are prominent examples of the placeholder view of assumptions.

4.1.1 The problem with refutations

In intuitionistic logic, *reductio ad absurdum* can be used to obtain negative sentences, or refutations. In such arguments, a contradiction (which in natural deduction systems is usually represented by an absurdity constant) is deduced from a collection of assumptions which are thereby shown to be jointly contradictory, or incompatible.

The task of explaining the validity of refutations becomes problematic when assumptions are considered placeholders for closed proofs and validity is explained as a constructive function from closed proofs of the assumptions to closed proofs of the conclusion, because the explanation then needs to appeal to proofs of contradictions. These proofs do not need to be actual proofs, but must be at least possible or conceivable if the explanation is to be at all comprehensible. Whether proofs of contradictions can be conceived, or what does it mean to conceive such things, is one of the questions that the advocates of the placeholder view have to deal with.

In some sense, the conundrum with *reductio ad absurdum* is reminiscent of a problem that Prawitz (1971, § IV.1.1) already dealt with in his first attempt at defining a proof-theoretic notion of validity. There, the problem was the vacuous validation of implications with an unprovable antecedent. Prawitz's solution was to reformulate the semantic clause for implication so as to consider extensions of the underlying atomic system where the antecedent would be provable.[1] However, the problem becomes much more prominent when dealing with contradictions, because our intuition is that they are not supposed to be provable under any circumstances whatsoever.

Moreover, for approaches based on the meaning of the logical constants as determined by their introduction rules, the clause for the absurdity constant becomes problematic, with suggestion ranging from infinitary rules (section 2.2.4) to the admission of inconsistent or trivial bases. The placeholder view of assumptions, especially in the case of subordinate hypothetical arguments for absurd conclusions, only accentuate these problems.

4.1.2 The primacy of assertion

Walking side by side with the placeholder view of assumptions is what I call *the primacy of assertion over other speech acts.* The rationale is that the speech act of assertion comes with a commitment on the part of the speaker to offer justifications for the asserted sentence and thus, in order to correctly assert the sentence, the speaker must be in possession of such justifications, or be able to produce them. In other words, in order to correctly assert a sentence, one needs to have a proof of the sentence.

From this picture emerges an approach to proof-theoretic semantics based on assertability conditions, with proofs acting as justifications associated with assertions. Here, another dogma discussed by Schroeder-Heister (2008, § 3) comes into play: *the transmission view of consequence.* But, in contrast with

1. Unfortunately, the amendment was still insufficient to avoid validation of classical inferences in the implication fragment (Sanz, Piecha and Schroeder-Heister 2014, § 4).

semantics based on truth conditions, instead of truth, it is *correct assertability* which is transmitted from premisses to conclusion in valid arguments. Or, if one prefers to talk about what makes an assertion correct, or justified, one can say that logical consequence transmits *proof* instead of *truth*. As a result, the approach assumes a distinctively epistemological character.

However, more complications related to hypothetical reasoning surface: it seems counterintuitive, to say the least, to hold that a speaker engaged in a hypothetical argument is committed to the *assertion* of either the assumptions or the conclusion of the argument. As a matter of fact, the speaker may even reject them and, provided the argument is indeed valid, her reasoning remains unassailable. In particular, the point reappears with renewed force when considered in the context of arguments that use *reductio ad absurdum*, since it would commit us to the possibility of correctly asserting absurdities.

One can appeal to a concept of *conditional assertion* to try and salvage the approach from such objections while preserving an unified explanatory model based primarily on assertion and proof. Thus, the conclusions of hypothetical arguments are taken not to be asserted outright but only under certain conditions. That is, the conclusions of hypothetical arguments are *conditionally asserted*. In terms of speech acts, however, it is not at all clear whether conditional assertion constitutes any assertion at all.

In particular, it is hard to defend that there is a commitment on the part of the speaker, explicit or implicit, to stand behind the assertion when it is a conditional one. Thus, the undogmatic and lively explanatory connection with our practices of offering and accepting reasons is significantly diminished and I wonder if it is worth to talk about assertions with little or no assertoric force in the first place. It could be argued that, in conditional assertions, the assertoric force does not apply to the conclusion but instead to the deductive relation between conclusion and the conditions (or assumptions). But, if transmission of assertability conditions is supposed to be the bedrock of deductive validity, are we not now presupposing what we set out to explain?

Furthermore, deductive validity interacts with other speech acts besides assertion. For instance, denial of the conclusion leads to denial of at least one of the assumptions. What makes assertion so special among the speech acts?

It seems to me that trying to explain deductive validity in terms of assertions and proofs is misguided. I am not trying to deny that deductive reasoning has epistemic importance or that deductive reasoning transmits evidence, or justification, from the assumptions to the conclusion. If there is a deductive relation between assumptions and conclusion, then, of course, the correct assertion of the assumptions would lead to the correct assertion of the conclusion and, similarly, if proof for the assumptions are provided then a proof for the conclusion is obtained. Rather, I contend that to *explain* deductive validity by reducing it to this transmission effect is to put the cart before the horse and confuse the cause with its effects, the disease with its symptoms.

4.2 Inadequacy to intuitionistic logic

Besides the conceptual difficulties already discussed, the correctness of standard proof-theoretic definitions can be questioned also on more technical grounds. Prawitz (1971, 2014) and Dummett (1975a, 1991) conjectured that proof-theoretic approaches to logical semantics would result in an intuitionistic, or constructive, notion of validity. It is relatively easy to show that intuitionistic logic is sound with respect to standard proof-theoretic semantics. On the other hand, since Prawitz's early proposals, completeness of intuitionistic logic remained a conjecture. However, many recent results indicate that this conjecture does not hold in the expected sense. Sandqvist (2009), for instance, have stated that some proof-theoretic approaches would in fact yield a constructive justification of classical logic. Sanz, Piecha and Schroeder-Heister (2014), in particular, argued that Prawitz's proposal results in a conflation of admissibility[2] and derivability, at least for a fragment of propositional logic. In this fragment, they have shown that Peirce's rule is admissible and thus valid according to Prawitz's definition.

Other counterexamples to completeness also appeared (Piecha, Sanz and Schroeder-Heister 2015). Some of them can be avoided by tinkering with the definitions. But I believe that the overall scenario (Piecha 2016) supports the conclusion that intuitionistic logic is not complete with respect to standard proof-theoretic semantics.

As I see it, the fact that standard proof-theoretic notions of validity validate classical or intermediary logics stronger than intuitionistic logic, even when only entrusted with strictly intuitionistic canons of reasoning, is a symptom of their conceptual inadequacy. The path to adequacy lies, first and foremost, in addressing the conceptual difficulties.

2. A rule r is *admissible* in a formal system S if, for every sentence A, whenever there is a closed proof of A in the system S extended by the rule r, then there is a closed proof of A in the system S without the rule r (assuming that r did not initially belonged to S).

5 Dummett's Justification Procedures and the Rejection of the Placeholder View of Assumptions

Dummett believed that the formal way of writing commonly employed in mathematics (with detached sequences of numbered definitions, theorems and proofs) was bad style in philosophy (Prawitz 2013).[1] Perhaps as a result of the writing style adopted, the proof-theoretic definitions of validity that he proposed are often viewed as imprecise and rudimentary.[2] They are, indeed, tentative, as are the various definitions proposed by Prawitz over the years. But if one takes the time to piece together precise formulations from the definitions scattered throughout the prose, what emerges is actually surprisingly rigorous and robust. In particular, one discovers notions of validity that are much more inspired by natural deduction and harmony than by the BHK interpretation.

5.1 BHK vs Gentzen

Proof-theoretic notions of validity have often been inspired by a mixture of ideas involving the BHK interpretation of the logical constants and Gentzen's informal remarks on the rules of natural deduction. In particular, the conception of validity underlying the placeholder view of assumptions is largely informed by the BHK interpretation of implication: an argument from A to B is valid if, and only if, every proof of A can be transformed into a proof of B. Yet, with its unqualified reference to proofs, this view is not immediately amenable to the recursive treatment required of semantic clauses and definitions (Prawitz 2007, § 2.1). In this context, Gentzen's ideas are often developed into a notion of canonical proof in order to achieve recursiveness for an approach primarily based on the BHK interpretation.

1. As can be seen from this text, I do not share Dummett's opinion on this issue, although I do try to keep formalisms to a minimum.
2. I sometimes see this view expressed in published materials and often hear it expressed in conversation.

On the other hand, the core of Gentzen's ideas are independent of the BHK interpretation. They are best represented by what became known as *proof-theoretic harmony* (section 2.3). Harmony, as a fundamental principle of normalisable natural deduction systems, applies equally well to deductions from assumptions as to the particular case of proofs (deductions from no assumptions). By appealing to harmony while at the same time avoiding the BHK interpretation and the placeholder view of assumptions, we can develop a more appropriate proof-theoretic notion of validity.

5.1.1 A more Gentzenian approach to validity

The introduction rules for a logical constant γ can be seen as an explanation of the *canonical use* of a sentence *as a conclusion* in a deductive argument (where, of course, γ is the sentence's main connective). This is achieved by exhibiting the conditions for obtaining a sentence $A \, \gamma \, B$ as a conclusion of an argument (where γ is a binary connective). In the paradigmatic case, these conditions are expressed in terms of the component sentences A and B.[3]

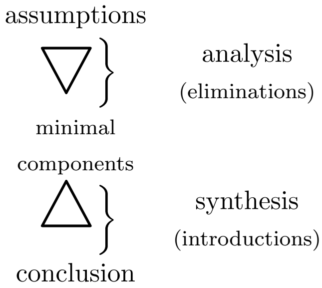
In an analogous manner, the elimination rules for a logical constant can be seen as an explanation of the *canonical use* of a sentence *as an assumption* in a deductive argument. This is accomplished by exhibiting the consequences that can be extracted from the sentence (as a major premiss of an elimination rule, possibly in the context of some minor premisses).

Thus, introduction and elimination rules stand for two distinct aspects of the deductive use of the logical constants. Harmony arises as a requirement of balance between those two aspects such that there is a equilibrium between what is required by the introduction rules and what consequences are extracted by the elimination rules. As a result, among other things, harmony guarantees that there is nothing to be gained by performing roundabout derivations where sentences are obtained by an introduction rule to be immediately after analysed by the corresponding elimination rule. Therefore, for a proper understanding of the deductive practice, it suffices to look at the collection of direct derivations, also known as normal derivations.

The normal derivations have a very perspicuous form (Prawitz 1965, § IV.2, Theorem 2). They are composed of (can be divided into) two parts: an analytic part, where the assumptions are analysed (destructed), and a synthetic part, where the conclusion is synthesized (constructed) from the components resulted

3. Notice that the conditions do not necessarily need to be expressed in terms of closed proofs of A and B, but can be expressed in terms of assumptions A and B or of arguments for A and B which may depend on other assumptions.

from the analysis.[4]



The equilibrium between introductions and eliminations suggest that, if we were to *supplement* the assumptions on top through a process of inversion by backward application of introductions, we would arrive at the minimal components required for the synthetic part. And, similarly, if we were to *complement* the conclusion by forward application of eliminations, we would arrive at the minimal components resulted from the analysis of the assumptions. Accordingly, the harmonious inferential behaviour of the logical constants has sometimes been expressed by pointing out that introductions and eliminations can be, in some sense, obtained from one another by inversion principles (section 2.3).

Gentzen's investigations into logical deduction can thus supply the basic pieces for a proof-theoretical notion of logical validity for arguments based on the inferential meaning conferred on the logical constants by either their introduction rules or their elimination rules. In particular with respect to the problems discussed in the previous section, the Gentzenian approach has the advantage of giving proper heed to assumptions and being fairly independent from *specific* speech acts.[5]

Gentzen's ideas suggest that, although a persistent dogma in much of the discussion around proof-theoretic semantics, the placeholder view of assumptions can be challenged from an authentic proof-theoretic perspective. In the next section, I revisit Dummett's justification procedures. I argue that, as a development of the Gentzenian approach just sketched, they afford a notion of proof-theoretic validity that incorporates assumptions in an essential way.[6]

4. In the general case, each of these parts can, of course, be empty.

5. For instance, deductive arguments can be used to show someone who denies the conclusion that she has to deny at least one of the assumptions. They can also be used to explore the consequences of a conjecture. These applications of deductive arguments align very well with the Gentzenian approach, but none of them necessarily involves anyone making any assertions.

6. It is important to notice that, although the Dummettian approach that I advocate rejects

I stay at the level of the core concepts, without going into rigorous definitions. Nonetheless, I hope that my explanations would be sufficient to give an overall idea of the relationship between the justification procedures (how they can be understood as emerging from a shared framework). This should provide a conceptual foundation for the detailed formal accounts in chapters 6 and 7.

5.2 An overview of Dummett's approach

Dummett (1991, Chapter 11–13) proposed two proof-theoretic justification procedures for logical laws which amount to definitions of logical validity for arguments. The "verificationist" procedure defines validity of arguments on the basis of introduction rules for logical constants and the "pragmatist" procedure defines validity of arguments on the basis of elimination rules for logical constants.[7]

These proof-theoretic justification procedures play an important role in Dummett's philosophical anti-realist programme. They are central pieces of his very detailed and elaborate argument for rejection of classical logic in favour of intuitionistic logic.[8] In particular, Dummett (1991, 1975a) has conjectured that proof-theoretic notions of validity would justify exactly intuitionistic logic.

Dummett's definitions of validity are based on canonical inference rules for the logical constants. These inference rules are thought to fix the meaning of the logical constants by displaying their canonical deductive use. They are, in Dummett's terminology, "self-justifying".

In contrast with some definitions found in the literature, Dummett's definitions are not based on semantic clauses for particular logical constants. Instead, he assumes that self-justifying rules are given. These self-justifying rules are

the placeholder view of assumptions, other dogmatic characteristics, like the unidirectional and global character of the semantics, remain unchallenged (Schroeder-Heister 2016, § 2.3 and 2.4).

7. I adopt the characterizations "verificationist" and "pragmatist" from Dummett. However, without denying their existence, I do not imply with the adoption of the terminology any connections outside the domain of logical validity. Therefore, I refer to verificationism and pragmatism just as markers to distinguish between approaches to validity based on introduction rules and elimination rules, respectively.

8. At the end of this argument, Dummett (1991, p. 300) writes: "We took notice of the problem what metalanguage is to be used in giving a semantic explanation of a logic to one whose logic is different. A metalanguage whose underlying logic is intuitionistic now appears a good candidate for the role, since its logical constants can be understood, and its logical laws acknowledged, without appeal to any semantic theory and with only a very general meaning-theoretical background. If that is not *the* right logic, at least it may serve as a medium by means of which to discuss other logics."

introduction rules in the context of the verificationist procedure, and elimination rules in the context of the pragmatist procedure. In both procedures, the definitions are stated irrespective of the particular constants or rules provided. Therefore, Dummett's definitions can, at least in principle, be applied without modification to different logics by providing the appropriate self-justifying rules for the logical constants.[9]

5.2.1 Core concepts

Both the verificationist and the pragmatist procedures can be seen as products of a basic common framework. The core notions of validity behind the justification procedures can be informally outlined as follows:

verificationism whenever the assumptions can be obtained in a canonical manner, the conclusion can also be obtained in a canonical manner.

pragmatism any consequence that can be drawn in a canonical manner from the conclusion can also be drawn in a canonical manner from the assumptions.

The expression "canonical manner" is an allusion to *canonical arguments.* As usual in proof-theoretic notions of validity, canonical arguments are the main ingredients of the justification procedures. An important feature, however, is that Dummett's canonical arguments are *not closed proofs*, but instead *may depend on assumptions.* Consequently, when precisely formulated, the definitions of validity must take into account the assumptions on which the canonical arguments depend.

The canonical arguments are composed *primarily* of canonical inferences. However, they cannot be required to be *entirely* composed of canonical inferences. They must allow for the possibility of *subordinate subarguments*, that is, subarguments cultivated under the support of additional assumptions (Dummett 1991, p. 260). These subordinate subarguments, when not already canonical arguments themselves, are *critical subarguments.* They are critical in the sense that the validity of the original canonical argument would recursively depend on their validity. This means, of course, that much care should be dispensed to guarantee that critical subarguments are of lower complexity than the original canonical argument.

9. For technical reasons, the procedures would certainly be restricted at the outset to logics with nice proof-theoretic properties (normalisation, subformula property and etc.), however there is no intrinsic technical limitation that confines it to intuitionistic logic or some specific formulation thereof.

In a verificationist context, critical subarguments are detected through the presence of assumption discharge. In a pragmatist context, they are detected through the presence of minor premisses. These signs indicate, in their respective contexts, when assumptions are being added.

Now, returning to the informal notions of validity, in the verificationist procedure, the means to evaluate the conditions under which the assumptions may be obtained in a canonical manner are provided by *supplementations.* They result from substitution of the assumptions with canonical arguments. In the pragmatist procedure, the means to evaluate what consequences can be drawn from the conclusion are provided by *complementations.* They result from substitution of the conclusion with canonical arguments.

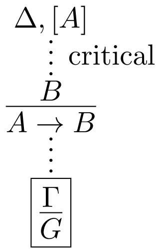
verificationism	**pragmatism**
canonical arguments	canonical arguments
(primarily introductions)	(primarily eliminations)
critical subarguments	critical subarguments
(revealed by assumption discharges)	(revealed by minor premises)
supplementation	complementation
(assumptions canonically unfolded)	(conclusion canonically unfolded)

Instead of as substitution operations, one can see the processes of supplementation and complementation more dynamically. The process of supplementation can be seen as the repeated backward application of introduction rules from the assumptions, thus growing the argument upwards, which is why Dummett also refers to the verificationist procedure as the *upwards justification procedure.* Similarly, the process of complementation can be seen as the repeated forward application of elimination rules to the conclusion of the argument as a major premiss, thus growing the argument downwards, which is why Dummett also refers to the pragmatist procedure as the *downwards justification procedure.*

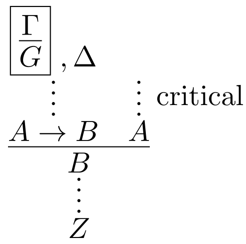
In order to appraise the validity of an argument from Γ to G, the verificationist procedure examines its supplementations and investigates whether a canonical argument for G can be attained under the same conditions. Since supplementations result from canonical arguments *for* Γ, they may depend on assumptions Δ (remember that canonical arguments may depend on assumptions). Then, the canonical argument for G may not depend on other assumptions besides Δ.

In an analogous manner, in order to appraise the validity of an argument from Γ to G, the pragmatist procedure examines the complementations and investigates whether a canonical argument for the conclusion of the complementation, say Z, can be attained under the same conditions Γ, Δ. Because complementations result from canonical arguments *from* G (as assumption and major premiss of elimination), they may depend on additional assumptions Δ required by minor premisses. Therefore, the canonical argument for Z may not depend on assumptions besides Γ, Δ.

Supplementation



Complementation



The canonical arguments used to supplement or complement may have critical subarguments. In the figures above, I indicate the form of possible supplementations and complementations of an illustrative argument from Γ to G. In the supplementation, the subargument from Δ, A to B may be critical. In the complementation, the subargument for the minor premiss A of $\to$E may be critical (for instance, if validly, but not canonically, obtained from assumptions in Γ, Δ).

6 Proof-Theoretic Validity Based on Introductions

For his notion of validity based on introductions rules, Dummett (1991) incorporates Prawitz's notion of basic systems (section 3.1) by means of what he called "boundary rules", that is, in my terminology, basic rules. Unfortunately, Dummett's imprecision on the matter of basic rules leaves some of his definitions open to interpretation. The original definitions affected are treated as informal characterisations which are latter turned into precise definitions.

6.1 Original definitions and characterisations

The characterisations and definitions are cosmetically rephrased, and adapted to the propositional case. It is perhaps useful to remind you that I adhere to the conventions, notation and terminology set out in section 2.1.

Definition 6.1.1. A sentence occurrence A is in the *main stem* of an argument Π if every sentence in the path from A to the conclusion of Π (inclusive) depends only on the assumptions of Π.[1]

The purpose of the concept of main stem is to keep track of discharged assumptions. Whether a sentence occurrence belongs to the main stem is perhaps best ascertained by climbing up from the root towards the leaves and examining the inferences for assumption discharge. Thus, subarguments whose conclusion depend on additional assumptions are identified. Notice that an application of $\to$I might not discharge any assumptions. Hence, whether or not the occurrence of B immediately above $A \to B$ is in the main stem depends on whether or not assumptions were actually discharged by the application of $\to$I.

Definition 6.1.2. A *critical subargument* of an argument Π is determined by a premiss outside the main stem of Π whose immediate consequence belongs to the main stem of Π.

1. Notice that the main stem of an argument may be composed of various distinct paths or branches.

The critical subarguments are, in other words, the largest subarguments that depend on more assumptions than the original argument. Consequently, when there are critical subarguments, some assumptions were discharged by the inference from the conclusion of the critical subargument to the immediate consequence belonging to the main stem. Perhaps the best way to identify critical subarguments is to climb the argument tree from the root conclusion towards the leaves, examining each path (branch): any sentence occurrence that depends on more assumptions than the conclusion (that is, any sentence occurrence that is a consequence of an inference discharging an assumption) determines a critical subargument.

Example 6.1.3. In the argument below, only the conclusion $(B \wedge C) \rightarrow (A \wedge B)$ is in the main stem. The subargument for $A \wedge B$ is critical.[2]

$$\dfrac{\dfrac{A \qquad \dfrac{[B \wedge C]}{B}}{A \wedge B}}{(B \wedge C) \rightarrow (A \wedge B)}$$

The next characterisations involve the notion of canonical argument and are essential to the definition of validity. As remarked (section 5.2), the verificationist procedure aims to define validity on the basis of a given collection of introduction rules. Besides the introduction rules, Dummett's notion of canonical argument also resort to basic rules. However, he is not sufficiently explicit about the general form of the basic rules and how they interact with the notion of canonical argument. As a result, Dummett's characterisations are not precise enough and should be regarded as provisional, although they do supply a general framework for the verificationist justification procedure. In section 6.2, I discuss the role played by basic rules and provide two possible interpretations for the reference to basic rules in the characterisation of canonical arguments.

Characterisation 6.1.4. An argument is *canonical* if the following three conditions hold:

(i) all its assumptions are atomic sentences

(ii) every atomic sentence in the main stem is either an assumption or is the consequence of a basic rule

2. When discussing this example, Dummett (1991, p. 263) doesn't follow his own definition. He claims that both the assumption A and the conclusion $(B \wedge C) \rightarrow (A \wedge B)$ are in the main stem. However, since the sentence $A \wedge B$ (which depends on $B \wedge C$) occurs in the path from A to the conclusion, A is not, after all, in the main stem. I am sure this is nothing but a small lapse rather than an indication of any subtlety in his definitions.

(iii) every complex sentence (not only the last one) in the main stem is obtained by means of an introduction rule

Characterisation 6.1.5. A *supplementation* of an argument is the result of replacing a valid canonical argument for each assumption. The replacing valid canonical arguments are called *supplements*.[3]

Example 6.1.6. Consider the argument

$$\frac{B \vee C}{(A \to B) \vee (A \to C)}$$

Any supplementations will take one of the forms below, where $a_1 \cdots a_n$ and $b_1 \cdots b_m$ are atomic assumptions.

$$\dfrac{\dfrac{\begin{matrix} a_1 \cdots a_n \\ \Pi_1 \\ B \end{matrix}}{B \vee C}}{(A \to B) \vee (A \to C)} \qquad \dfrac{\dfrac{\begin{matrix} b_1 \cdots b_m \\ \Pi_2 \\ C \end{matrix}}{B \vee C}}{(A \to B) \vee (A \to C)}$$

Characterisation 6.1.7. A canonical argument is *valid* if all its critical subarguments are valid.

Characterisation 6.1.8. An argument is *valid* if any supplementation can be effectively transformed into a valid canonical argument for the same conclusion from, at most, the same assumptions.

Example 6.1.9. Consider the argument

$$\frac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C)}$$

By characterisation 6.1.8, it is valid provided any supplementation can be transformed into a valid canonical argument for $(A \wedge B) \vee (A \wedge C)$ from, at most, the same assumptions. Now, consider supplements for the assumption $A \wedge (B \vee C)$

$$\dfrac{\begin{matrix} a_1 \cdots a_n \\ \Pi_1 \\ A \end{matrix} \qquad \dfrac{\begin{matrix} b_1 \cdots b_m \\ \Pi_2 \\ B \end{matrix}}{B \vee C}}{A \wedge (B \vee C)}$$

3. Originally, Dummett restricted the replacement only to complex assumptions. This restriction, although conceptually helpful, is formally dispensable because, by characterisation 6.1.4, atomic assumptions are themselves canonical arguments.

From the arguments Π_1 for A and Π_2 for B, a valid canonical argument for $(A \wedge B) \vee (A \wedge C)$ can be obtained by applications of introduction rules.

$$\dfrac{\dfrac{\begin{matrix} a_1 \cdots a_n \\ \Pi_1 \\ A \end{matrix} \quad \begin{matrix} b_1 \cdots b_m \\ \Pi_2 \\ B \end{matrix}}{A \wedge B}}{(A \wedge B) \vee (A \wedge C)}$$

Transformations of supplementations are the main elements in the verificationist justification procedure. The fact that supplementations are obtained by replacing valid canonical arguments for the assumptions betrays an, at least partial, adherence to the placeholder view of assumptions. However, Dummett's commitment to the placeholder view is not exactly unwavering, because he allows canonical arguments to depend on atomic assumptions. As an immediate consequence, for every sentence A, there is a canonical argument for A, built from atomic subsentences of A by a series of introduction rules.[4]

6.2 Verificationist validity and basic rules

Characterisation 6.1.4 is the only place where explicit reference to basic rules is made. However, because of their interconnection, the characterisations that follow also depend on bases. As a result, the concept of verificationist validity can not be properly and unambiguously understood without a careful examination of the role played by basic rules. Dummett himself does not offer a detailed discussion of basic rules. His book contains what seem to be conflicting ideas and intuitions on the matter.

6.2.1 Bases and canonical arguments

The characterisations in section 6.1 may be regarded as relative to a previously fixed basis B. In this case, the phrase "in a given basis B" should be added to clause (ii) of characterisation 6.1.4. Consequently, any reference to canonicity should be understood with respect to such a given basis B. Notice, in particular, that supplementations should be given in B because supplements are canonical arguments.

This interpretation makes the concept of validity dependent on the basis B under consideration. For this reason, I call it the *dependent interpretation*. In

4. As a limiting case, there are canonical arguments for atomic sentences by an empty series of introduction rules.

a discussion of justification procedures of the second grade, Dummett (1991, p. 254) seems to assume the dependent interpretation:

> We assume that we are given certain rules of inference, which we recognize as valid, for deriving atomic sentences from one or more other atomic sentences; we may call these 'boundary rules'. We now define a 'canonical argument' to be one in which no initial premiss is a complex sentence (no complex sentence stands at a topmost node) and in which all the transitions are in accordance either with one of the boundary rules [...]

Of course, Dummett's primary concern is with the justification of logical laws and it is reasonable to expect that the validity of logical laws should not depend on particular features of bases. Dummett (1991, p. 273, my emphasis) says:

> We originally admitted, as occurring within deductive proofs of the kind with which we are concerned, boundary rules allowing the inference of an atomic conclusion from atomic premisses: these were, of necessity, left unspecified. Our original intention was that the boundary rules should be deductively valid. If we now include among them principles of non-deductive (and therefore fallible) inference, this will have the effect that a 'valid' argument, even if canonical, may have true initial premisses but a false final conclusion. *It will obviously not affect the justification procedure, however, as a means of determining the validity of logical laws.*[5]

In order to define logical validity under the dependent interpretation, some kind of generalization with respect to bases is required. The issue is going to be explored in section 6.4.

On the other hand, a basic rule is not recognized as such because it belongs to a basis. It is recognized as basic because it has a certain general form. Hence, the reference to basic rules in item (ii) of characterisation 6.1.4 could be interpreted as standing for a generic formulation for inferences of basic form. The characterisation could thus become independent of any particular basis. I call this approach the *independent interpretation.* In section 6.3, the independent interpretation is developed into a notion of validability which is then straightforwardly used to define logical validity. Logical principles justified in accordance with it do not depend on any specific bases.

5. This quotation is extracted from a later chapter, after Dummett had already presented his verificationist justification procedure.

6.2.2 The general form of basic rules

An important question regarding the general form of basic rules is whether or not they should be allowed to discharge hypotheses. In his first attempt to define a proof-theoretic notion of validity, Prawitz (1971) considered the validity of atomic sentences by means of production rules, i. e. basic rules that do not discharge hypotheses. Since then, the standard proof-theoretic approach is to use a system of production rules in order to give the assertability conditions for atomic sentences.

On the other hand, Sanz, Piecha and Schroeder-Heister (2014, §,5.1) observed that a framework where basic rules discharge assumptions could avoid the validation of classical laws in the implication fragment. Moreover, Sandqvist (2015) proposed a proof-theoretic semantics where basic rules discharge assumptions and proved completeness of propositional intuitionistic logic with respect to it. However, as remarked by Piecha, Sanz and Schroeder-Heister (2015, §,7), the legitimacy of basic rules that discharge assumptions is very problematic because they amount to the admission of implication into the premisses of basic rules. The contention is that this effectively enables logical content to be smuggled into the basis. In principle, Dummett (1991, p. 255) does not seem to be opposed to the idea:

> The need to allow for the application of boundary rules is not as yet apparent but evidently can do no harm: they might be rules governing either non-logical expressions or logical constants not in the given set.

From the quotation, it is not clear if discharge of assumptions in basic rules is permissible, but, since basic rules could conceivably cover logical constants, the way for this kind of basic rules seems to be open. Nevertheless, Dummett (1991, p. 261) seems to tacitly assume that basic rules are production rules. While discussing the danger of circularity in the definition of validity, he presupposes that critical subarguments only occur in arguments for complex conclusions:

> It is important to notice that a sentence A standing immediately below the conclusion C of a critical subargument of a canonical argument must be of higher logical complexity than either the conclusion or the premisses of that subargument. This holds good of C because A, being a closed sentence in the main stem, must be derived by an application of one of the introduction rules, of which C must accordingly be one of the premisses; by the complexity condition on the introduction rules, A must be of higher logical

complexity than any of its premisses. The premisses of the subargument must either be initial premisses of the entire argument, in which case they are atomic, or be hypotheses discharged by the introduction rule, in which case they must again be of lower logical complexity than A.

Notice that, if basic rules are allowed to discharge assumptions, basic arguments (that is, arguments containing only basic rules) may have critical subarguments (definitions 6.1.1 and 6.1.2). Therefore, when basic rules discharge assumptions, circularity could become a serious problem. I do not examine this problem because, for my purposes, it suffices to consider bases without discharge. Nonetheless, some results below do not depend on the issue of whether or not to allow discharge in basic rules. These cases are explicitly mentioned.

6.3 Independent interpretation

In the independent interpretation, basic rules are recognized by their general form. However, a restriction must be imposed on the transformations described in characterisation 6.1.8: they cannot appeal to new basic rules. The rationale is that, whatever basic rules appear in the supplements, a transformation should not use more than those rules in order to construct the canonical argument for the conclusion. The restriction is similar to the one Dummett already imposes on assumptions. By adopting the independent interpretation, validity of basic arguments may be left unspecified. The notion characterised under the independent interpretation is called *validability*, instead of validity, in order to call attention to the fact that its applicability to basic arguments is purely formal and do not attest to the legitimacy of its basic rules as coherent principles of reasoning (something that may be expected of reasonable atomic systems). The characterisations of section 6.1 can now be made into precise definitions. The definitions of the notions of main stem and critical subargument are reproduced without changes.

Definition 6.3.1. A sentence occurrence A is in the *main stem* of an argument Π if every sentence in the path from A to the conclusion of Π (inclusive) depends only on the assumptions of Π.[6]

6. Notice that the main stem of an argument may be composed of various distinct paths or branches.

Definition 6.3.2. A *critical subargument* of an argument Π is determined by a premiss outside the main stem of Π whose immediate consequence belongs to the main stem of Π.

Definition 6.3.3. An argument is *I-canonical* (canonical in the context of the independent interpretation), if the following three conditions hold:

(i) all its assumptions are atomic sentences

(ii) every atomic sentence in the main stem is either an assumption or is the consequence of a basic rule
$$\frac{a_1 \cdots a_n}{b}$$
where $a_1 \cdots a_n$ and b are atomic sentences

(iii) every complex sentence in the main stem is obtained by means of an introduction rule

Definition 6.3.4. An *I-supplementation* of an argument is the result of replacing validable I-canonical arguments for each assumption. The validable I-canonical arguments are called *I-supplements.*

Definition 6.3.5. An I-canonical argument is *validable* if all its critical subarguments are validable.

Definition 6.3.6. An argument is *validable* if any I-supplementation can be effectively transformed into a validable I-canonical argument for the same conclusion containing no additional assumptions and no additional basic rules.

The definitions are interconnected. In particular, definitions 6.3.4 to 6.3.6 simultaneously define the concepts of *I-supplementation*, *validable I-canonical argument* and *validable argument.* The definitions are better understood recursively, where validable arguments are defined in terms of validable I-canonical arguments which in turn are defined in terms of validable critical subarguments. This may raise concerns about circularity in the definition of validability. In particular, the complexity must decrease from validable arguments to validable critical subarguments in order for the recursion to be well-founded.

Definition 6.3.7. The degree of an argument $\langle \Gamma, G \rangle$ is the maximum among the degrees of the sentences in Γ and the conclusion G.

Theorem 6.3.1 (Dummett 1991)**.** *The definition of validable argument is well-founded.*

Proof. Let $\langle \Gamma, G \rangle$ be an argument of degree n. Suppose that $\langle \Gamma, G \rangle$ is in canonical form. Then, its validability depends only on the validability of its critical subarguments, which, by definitions 6.3.1 to 6.3.3, are all of lower degree.[7] Now, suppose that $\langle \Gamma, G \rangle$ is not in canonical form. Then, by definition 6.3.6, in order to evaluate its validability, we have to consider transformations from I-supplements for the assumptions Γ into validable I-canonical arguments for G. By definition 6.3.3, these I-canonical arguments are of degree n, at most, and their critical subarguments are, consequently, of strictly lower degree. □

Definition 6.3.8. An argument is *LI-valid* (logically valid under the independent interpretation) when it is validable and it contains no applications of basic rules.

Definition 6.3.9. An argument is *valid in a basis* B under the independent interpretation when it is validable and all the basic rules used in the argument belong to B.

Theorem 6.3.2. *Let a and b be atomic sentences. Then atomic Peirce's rule*

$$\frac{(a \to b) \to a}{a}$$

is LI-valid.

Proof. I show that any I-supplementation can be effectively transformed into a validable I-canonical argument for the conclusion depending on the same assumptions and no additional basic rule. Suppose Π_1 is an I-supplementation depending on assumptions $x_1 \cdots x_n$, as specified by definition 6.3.4. Thus, by definition 6.3.3, the conclusion of the I-supplement was obtained by an application of →I as shown below.

$$\cfrac{\cfrac{\begin{matrix} x_1 \cdots x_n \\ \Pi_2 \\ a \end{matrix}}{(a \to b) \to a}}{a} \qquad (\Pi_1)$$

There are two possibilities: either (1) the conclusion a of Π_2 is in the main stem and we already have a validable I-canonical argument for a from the same assumptions and no additional basic rule, by item (ii) of definition 6.3.3, or (2) a is not in the main stem and Π_2 is a critical subargument, by definition 6.3.2.

7. Of course, under the assumption that the introduction rules comply with a complexity condition, as formulated by Dummett (1991, p. 258). In particular, premisses of introduction rules must be subsentences of their consequence.

In the second case, by definition 6.3.2, a depends on additional assumptions besides $x_1 \cdots x_n$. These assumptions were discharged by an application of →I whose conclusion is $(a \to b) \to a$. Therefore, they can only be of the form $a \to b$.

$$\begin{array}{c} a \to b, x_1 \cdots x_n \\ \Pi_2 \\ a \end{array}$$

By definition 6.3.6, the critical subargument Π_2 is *validable*. From the validability of Π_2, I show how to obtain a validable I-canonical argument for a from atomic assumptions $x_1 \cdots x_n$ and no additional basic rules. Because Π_2 is validable, we have a procedure to effectively transform *any* I-supplementation Π_3 into a validable I-canonical argument for a from assumptions $x_1 \cdots x_n$ and, possibly, other assumptions $y_1 \cdots y_m$.

$$\begin{array}{c} \dfrac{\begin{array}{c} y_1 \cdots y_m \\ \Pi_4 \\ b \end{array}}{a \to b}, \; x_1 \cdots x_n \\ \Pi_2 \\ a \end{array} \qquad (\Pi_3)$$

In particular, consider the following I-supplementation obtained by substitution of I-supplements for assumptions of Π_2.

$$\begin{array}{c} \dfrac{\dfrac{[a]}{b}}{a \to b}, x_1 \cdots x_n \\ \Pi_2 \\ a \end{array} \qquad (\Pi_{3_{\varphi/\psi}})$$

This I-supplementation (containing an additional basic rule from a to b) is then transformed into a validable I-canonical argument Π_7 from assumptions $x_1 \cdots x_n$ only. By definition 6.3.3, this validable I-canonical argument proceeds solely by basic rules. Examining Π_7, if the additional rule is not used, then we already have a validable I-canonical argument from assumptions $\alpha_1 \cdots \alpha_n$ and no additional rules. Otherwise, if the additional rule is used, we take its first application as depicted below.

$$\begin{array}{c} \alpha_1 \cdots \alpha_n \\ \Pi_5 \\ \dfrac{\varphi}{\psi} \\ \Pi_6 \\ \varphi \end{array} \qquad (\Pi_7)$$

Since the rule does not occur in the subargument Π_5, we obtain the required valid I-canonical argument for a from assumptions $x_1 \cdots x_n$ and no additional basic rules.[8] □

Corollary 6.3.3. *Propositional intuitionistic logic is incomplete under the independent interpretation.*

Once all atomic instances of Peirce's rule are shown to be valid, it is possible to generalize the result for a fragment of the language without disjunction. This fragment is powerful enough to account for all valid propositional classical reasonings (with the other constants being defined in terms of implication, conjunction, and negation).

Theorem 6.3.4. *Let A and B be any sentences. Assuming soundness of* $\mathsf{N}_{\to,\wedge,\bot}$, *Peirce's rule*

$$\frac{(A \to B) \to A}{A}$$

is LI-valid for the respective fragment.

Proof. By recursion on the degree of B and A, with theorem 6.3.2 as the base case. For the recursion on B, I only show the case for $\wedge$, where $B = D \wedge E$. The case for $\to$ is analogous.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A \to (D \wedge E)) \to A \qquad \dfrac{\dfrac{\dfrac{[A \to D]^{(2)} \quad [A]^{(3)}}{D} \qquad \dfrac{[A \to E]^{(1)} \quad [A]^{(3)}}{E}}{D \wedge E}}{A \to (D \wedge E)}\,{\scriptstyle (3)}}{A}}{(A \to D) \to A}\,{\scriptstyle (2)}}{A}\,\text{Peirce's rule}}{(A \to E) \to A}\,{\scriptstyle (1)}}{A}\,\text{Peirce's rule}$$

Now, for the recursion on A, I show the case for $\to$, where $A = D \to E$. Again,

8. The proof depends on the restriction to basic rules without discharge. In particular, it depends on the absence of discharges among the rules in Π_6.

the other cases are similar.

$$
\dfrac{\dfrac{\dfrac{\dfrac{((D \to E) \to B) \to (D \to E) \qquad \dfrac{\dfrac{[E \to B]^{(2)} \qquad \dfrac{[D \to E]^{(3)} \quad [D]^{(1)}}{E}}{B}}{(D \to E) \to B}\,(3)}{D \to E} \qquad [D]^{(1)}}{E}}{(E \to B) \to E}\,(2)}{\dfrac{E}{D \to E}\,(1)}\ \text{Peirce's rule}
$$

□

It is reasonable that proof-theoretic definitions could validate classical logic with respect to a suitable, classically designed, collection of introduction rules. Notice, however, that the introduction rules for $\mathsf{N}_{\to,\wedge,\bot}$ incorporate no classical principles of reasoning. Hence, theorem 6.3.4 reveals a strong inadequacy of the definitions under the independent interpretation.

6.4 Dependent interpretation

Under the dependent interpretation, the primary notion of validity becomes relative to a fixed basis B. In other words, the supplementation of arguments and the valid canonical arguments are all given in a fixed basis B. I formulate definitions that correspond to the characterisations in section 6.1, making explicit reference to a basis B. The definitions of the notions of main stem and critical subargument are again reproduced without changes.

Definition 6.4.1. A sentence occurrence A is in the *main stem* of an argument Π if every sentence in the path from A to the conclusion of Π (inclusive) depends only on the assumptions of Π.[9]

Definition 6.4.2. A *critical subargument* of an argument Π is determined by a premiss outside the main stem of Π whose immediate consequence belongs to the main stem of Π.

Definition 6.4.3. An argument is *canonical in* B if the following three conditions hold:

9. Notice that the main stem of an argument may be composed of various distinct paths or branches.

(i) all its assumptions are atomic sentences

(ii) every atomic sentence in the main stem is either an assumption or is obtained by a rule in B

(iii) every complex sentence in the main stem is obtained by means of an introduction rule

Definition 6.4.4. A *supplementation in* B of an argument is the result of replacing valid canonical arguments in B for each assumption. The valid canonical arguments added are called *supplements.*

Definition 6.4.5. A canonical argument is *valid in* B if all its critical subarguments are valid in B.

Definition 6.4.6. An argument is *valid in a basis* B if any supplementation in B can be effectively transformed into a valid canonical argument in B for the same conclusion from, at most, the same assumptions.

The notion of validity in definition 6.4.6 is relative to a fixed basis B. The notion of logical validity, on the other hand, must maintain a certain independence of bases and basic rules. It could be presumed that logical validity is captured by the case of the empty basis. As it turns out, however, the empty basis is not adequate for the task.

Theorem 6.4.1 (Goldfarb (1998) 2016)**.** *Let a and b be distinct atomic sentences. The argument*

$$\frac{a \to b}{b}$$

is valid in the empty basis.

Proof. By definition 6.4.6, the argument is valid if any supplementation of $a \to b$ can be transformed into a valid canonical argument for b from, at most, the same assumptions. There are no means of obtaining atomic sentences from other atomic sentences in the empty basis. Therefore, every possible supplementation involves the assumption of b. Among them, the simplest one is:

$$\cfrac{\cfrac{b}{a \to b}}{b}$$

By definition 6.4.3, the assumption b alone is a valid canonical argument for b in any base. □

Corollary 6.4.2 (Goldfarb (1998) 2016)**.** *Validity of arguments is not preserved through monotonic extensions of the basis.*

Proof. By theorem 6.4.1, the argument from $a \to b$ to conclusion b is valid in the empty basis, when a and b are distinct atomic sentences. However, it is not valid in the immediate extension of the empty basis containing a rule enabling the inference of b from a. □

Consequently, bases cannot be correctly interpreted as knowledge bases monotonically extended, at least for the dependent interpretation. In particular, the empty basis is better understood as representing *knowledge of the absence* of any valid inferences among atomic sentences rather than *the absence of knowledge* about the valid inferences among atomic sentences. This makes the empty basis a particular basis, not a general one, as is expected in Prawitz's framework of monotonic extensions (chapter 3), where the empty basis can act as a surrogate for an arbitrary basis. The remainder alternative is to define logical validity for the dependent interpretation as validity in all bases.

Definition 6.4.7. An argument is *LD-valid* (logically valid under the dependent interpretation) if it is valid in all bases.

The dependent notion of logical validity avoids validation of classical logic through the same means available under the independent interpretation. In this sense, the dependent interpretation can be considered more robust than the independent interpretation.

Example 6.4.8. I show that atomic Peirce's rule is not valid in some basis. Consider a basis B with a single rule

$$\frac{b}{a}$$

The argument from $a \to b$ to b is valid in B (theorem 6.4.1). It is impossible to transform the following valid canonical argument in B from no assumptions into a valid canonical argument for a in B from no assumptions.

$$\cfrac{\cfrac{\cfrac{[a \to b]}{b}\ \text{valid in } \mathsf{B}}{a}}{(a \to b) \to a}$$

Notwithstanding its success in avoiding validation of Peirce's rule, the dependent interpretation is still not an adequate semantics for propositional intuitionistic logic.

Theorem 6.4.3 (Goldfarb (1998) 2016)**.** *For any atomic a, the argument*

$$\frac{a \to (B \vee C)}{(a \to B) \vee (a \to C)}$$

is LD-valid.

Proof. Consider an arbitrary basis. In a supplement for $a \to (B \vee C)$ from assumptions $x_1 \cdots x_n$, with $B \vee C$ not in the main stem,[10] the critical subargument Π_1 must be valid, by definition 6.4.4.

$$\begin{array}{c} a, x_1 \cdots x_n \\ \Pi_1 \\ B \vee C \end{array}$$

By definition 6.4.3, since a is atomic, it can be its own supplement. Therefore, by definition 6.4.6, any supplementations of Π_1 can be transformed into a valid canonical argument for $B \vee C$ depending solely on a and $x_1 \cdots x_n$. The last step on this canonical argument is $\vee$I from either B or C. In both cases, we obtain a canonical argument for $(a \to B) \vee (a \to C)$ from $x_1 \cdots x_n$ by $\to$I and $\vee$I. □

Corollary 6.4.4. *Propositional intuitionistic logic is incomplete under the dependent interpretation.*

6.5 Discussion

The definition of validity based on introduction rules proposed by Dummett (1991) has noteworthy differences from other proof-theoretic definitions in the literature. In particular, Dummett's verificationist procedure admits assumptions in valid canonical arguments and, consequently, do not explicitly rely on bases being monotonically extended. His characterisations, however, are not precise enough when it comes to the role played by basic rules.

Based on some passages from his work, I proposed two interpretations of the verificationist justification procedure with respect to bases. In both interpretations, however, the verificationist justification procedure is not adequate as a semantics for propositional intuitionistic logic, when the standard introduction rules are considered.

In the independent interpretation of Dummett's procedure, atomic Peirce's rule is valid. For a suitable fragment, the validity of Peirce's rule can be

10. The case with $B \vee C$ in the main stem is trivial.

generalized for complex sentences. In contrast with the results for standard proof-theoretic semantics (Sanz, Piecha and Schroeder-Heister 2014; Piecha, Sanz and Schroeder-Heister 2015), this result does not explicitly rely on bases being monotonically extended. If the independent interpretation is accepted, theorem 6.3.4 frustrates the expectation, expressed by Dummett (1991, p. 270) himself, that proof-theoretic validity provides justification only for constructive reasonings.

On the other hand, the dependent notion of validity has the interesting and surprising property that it is not conservative over monotonic extensions of bases (corollary 6.4.2). It avoids validation of Peirce's rule. Nonetheless, theorem 6.4.3 indicates that there are still problems to be solved.

6.5.1 A deceptive amendment

The counterexamples of theorems 6.3.2 and 6.4.3 are restricted to at least some of the sentences in the argument being atomic. They are not effective for arbitrary complex sentences. Thus, the counterexamples could be easily blocked if schematicity is imposed, for instance, by superposing the original notion of logical validity with a schematic substitution requirement (Goldfarb [1998] 2016, § 3).

I agree with Goldfarb ([1998] 2016, § 4) that such purported solutions are conceptually unsatisfactory. First, schematicity should be an inherent and immanent property of the semantics, not a feature to be imposed from the outside. Second, on the conceptual level, such an amendment amounts to enabling atomic sentences to emulate arbitrary complex sentences and basic rules to emulate elimination rules. Although this could grant extensional adequacy with respect to intuitionistic logic, it is conceptually *ad hoc* and hardly defensible philosophically.

7 Proof-Theoretic Validity Based on Eliminations

The elimination rules express the canonical deductive use of sentences as assumptions in an argument (section 5.1.1). Therefore, they are particularly favourable as foundation for a notion of validity where hypothetical reasoning (reasoning from assumptions) is taken as primary, in contrast with the placeholder view of assumptions that informs standard proof-theoretic notions based on the introduction rules (section 4.1).

Although definitions are formulated for elimination rules in general (section 5.2), this chapter is particullarly concerned with their application to the standard elimination rules for the system N of propositional intuitionistic logic. Conventions, notation and terminology were described in section 2.1.

7.1 Preamble

7.1.1 Elimination rules and related notions

In an elimination rule for a logical constant, exactly one premiss of the rule is required to have that constant as main logical operator. This premiss is called the *major premiss*, and all others, if there are any others, are called *minor premisses*.

In an elimination rule, a minor premiss is *vertical* if the same sentence figures as both minor premiss and consequence of the rule, otherwise it is called *horizontal*. An elimination rule is a *vertical rule* if at least one of its minor premisses is vertical *and* it allows the discharge of assumptions in the subarguments for its vertical minor premisses. I assume that every application of vertical elimination rules do in fact discharge the assumptions as indicated by the rule.[1] Elimination rules that are not vertical are called *reductive*.

Notice that this terminology applies, first and foremost, to the *schematic rules* and is employed in the context of particular inferences only insofar as

1. Otherwise, the application of the elimination rule would be superfluous (Prawitz 1965, § IV.1). This is in line with Dummett (1991, p. 283).

they are recognised as applications of the rules in question. For example, the minor premiss A in

$$\frac{A \to A \quad A}{A}$$

is by no means vertical, but horizontal, despite the fact that A occurs both as minor premiss and consequence in this particular inference. As examples of vertical rules, there are the standard elimination rule for disjunction ∨E and the alternative generalized elimination rule for implication →GE.

$$\frac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \vee\mathrm{E} \qquad\qquad \frac{A \to B \quad A \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C} \to\mathrm{GE}$$

The first minor premiss of →GE is horizontal and the second one is vertical. Since it takes only one vertical minor premiss to make a rule vertical, →GE is vertical. In contrast, the standard elimination rule for implication →E, as well as ∧E and ⊥E, are reductive rules.

7.1.2 Normal derivations

This subsection recollects some results and fixes some terminology regarding normal derivations in propositional intuitionistic logic. The results are restated mainly for your convenience and to avoid confusion resulting from conflicting terminology. For detailed proofs, please resort to Prawitz's monographs (Prawitz 1965, 1971).

Definition 7.1.1. A *track*[2] in an argument Π is a sequence of *sentence occurrences* $A_1, \cdots A_n$ such that:

(i) A_1 is a leaf in Π that is not discharged by an application of a vertical elimination rule

(ii) A_i, for each $i < n$, is not a horizontal minor premiss of an application of an elimination rule, and either (1) A_i is not a major premiss of a vertical rule and A_{i+1} is the sentence immediately below A, or (2) A_i is a major premiss of a vertical rule and A_{i+1} is an assumption discharged by the respective application of the rule

2. The notion of "track" is obviously a slight adaptation of Prawitz's notion of "path" (Prawitz 1965, § IV.2). Since the term "path" is already reserved for a different concept, the term "track" is used in order to avoid confusion.

(iii) A_n is either a horizontal minor premiss or the conclusion of Π

As a sequence of occurrences of sentences, a track can also be divided into *segments*; they conflate repeated consecutive occurrences of the same sentence that arise from applications of vertical elimination rules. If the first sentence in a track is an assumption of the argument, the track is *open* in the argument, otherwise it is *closed*.

The tracks in an argument can be assigned an order. The lowest order is assigned to tracks whose last sentence occurrence is the conclusion of the argument; they are called *main tracks*. The order then increases progressively through horizontal minor premisses. The major premiss of the horizontal rule belongs to a *parent track* which, of course, can have other *children tracks* sharing an immediately higher order than their parents. The last sentence occurrence in a track determines a subargument, or subderivation, whose order is the order of the track. The progeny relationship between tracks can be naturally extended to cover them.

Theorem 7.1.1. *Let τ be a track in a normal intuitionistic derivation and let $\sigma_1 \cdots \sigma_n$ be the corresponding sequence of segments in τ. Then there is a segment σ_i, called the* base segment *in τ, which separates two (possibly empty) parts of τ, called the* analytic part *and the* synthetic part *of τ, such that:*

(i) for each σ_j in the analytic part, σ_j is a major premiss of an elimination rule and the sentence occurring in σ_{j+1} is a subsentence of the one occurring in σ_j

(ii) the base segment σ_i is a premiss of an introduction rule or of $\bot E$, provided $i \neq n$

(iii) for each σ_j in the synthetic part, except the last one, σ_j is a premiss of an introduction rule and the sentence occurring in σ_j is a subsentence of the one occurring in σ_{j+1}

Definition 7.1.2. The subsentences of a sentence A are classified as *positive* or *negative* as follows:

- A is a positive subsentence of A
- if $B \wedge C$ or $B \vee C$ are positive (resp. negative) subsentences of A, then B and C are positive (resp. negative) subsentences of A
- if $B \to C$ is a positive (resp. negative) subsentence of A, then B is a negative (resp. positive) subsentence of A and C is a positive (resp. negative) subsentence of A

Definition 7.1.3. A sentence A is an *assumption component* (resp. *conclusion component*) of an argument $\langle \Gamma, G \rangle$ when A is a positive (resp. negative) subsentence of some assumption in Γ, or a negative (resp. positive) subsentence of the conclusion G.

The notions defined above can be naturally extended to cover segments, whereby a segment σ_j is a subsegment of a segment σ_i if the sentence occurring in σ_j is a subsentence of the sentence occurring in σ_i.

Theorem 7.1.2. *Let $\sigma_1 \cdots \sigma_n$ be a track in a normal derivation of G from Γ. It holds that:*

(i) every segment occurring in the analytic part is an assumption component of $\langle \Gamma, G \rangle$ and subsegment of σ_1

(ii) the base segment σ_i is an assumption component of $\langle \Gamma, G \rangle$ and a subsegment of σ_1; also, if different from $\bot$, σ_i is a conclusion component of $\langle \Gamma, G \rangle$ and a subsegment of σ_n

(iii) every segment occurring in the synthetic part is a conclusion component of $\langle \Gamma, G \rangle$ and a subsegment of σ_n

7.1.3 Succinct derivations

I build up on results and definitions of the previous subsection in order to provide a complexity measure for subderivations in normal derivations. This complexity measure is formulated solely in terms of the assumptions and conclusion and, therefore, can be applied to arbitrary arguments and their subarguments. It is later employed in the definition of validity, particularly definitions 7.2.6 and 7.2.7. Theorem 7.1.4 guarantees that the complexity can be required to decrease with the order of the (critical) subarguments without losing soundness (section 7.3.2).

Definition 7.1.4. A *negative assumption* (resp. *conclusion*) *component* of an argument $\langle \Gamma, G \rangle$ is a negative subsentence of G (resp. some sentence in Γ). Analogously, a *positive conclusion* (resp. *assumption*) *component* of an argument $\langle \Gamma, G \rangle$ is a positive subsentence of G (resp. some sentence in Γ).

The negative and positive partitions introduced in definition 7.1.4 make out the components in definition 7.1.3. The assumption and conclusion components afford a coarse summary of the sentences occurring in the elimination and introduction parts of tracks in a normal derivation on the basis of its assumptions and conclusion. Their partition into negative and positive constituents yields a somewhat finer distinction.

Definition 7.1.5. A *distinctive assumption* (resp. *conclusion*) *component* of an argument is either a negative assumption (resp. conclusion) component or an assumption (resp. the conclusion).

The distinctive assumption components of an argument correspond intuitively to the first sentences (leaves) of its tracks. They are either assumptions or, possibly, leaves discharged by applications of →I and, consequently, negative subsentences of the conclusion. Notice that the negative assumption components represent, in a sense, a *potential* discharge. Whether an actual leaf was discharged would depend on the particular derivation.

The distinctive conclusion components correspond intuitively to last sentences (roots) of tracks. They include the conclusion of main tracks and, possibly, conclusions of their descendant tracks. More precisely, provided a parent track is open, the conclusion of a child track is a negative subsentence of its assumption.

Definition 7.1.6 (Dershowitz and Manna 1979)**.** The *degree of a (finite) collection of sentences* Δ_j is lower than the degree of a collection of sentences Δ_i ($\Delta_j < \Delta_i$) if, and only if, Δ_j results from Δ_i by replacing one or more sentences with a finite collection of sentences of lower degree.

Lemma 7.1.3. *In a normal derivation, the degree of the collection of distinctive conclusion components never increases with the order of subderivations. In particular, if a parent track is closed, then the collection of distinctive conclusion components of child subderivations have lower degree.*

Proof. Consider derivations Π_i and Π_j, where Π_j is a child of Π_i. I show that any distinctive conclusion component of Π_j is either itself a distinctive conclusion component of Π_i, or is replaced in Π_i with a distinctive conclusion component of higher degree. By definition 7.1.5, a distinctive conclusion component of Π_j is either the conclusion of Π_j or a negative conclusion component of Π_j. First, consider the negative conclusion components of Π_j. By definition 7.1.4, they also belong to the distinctive conclusion components of Π_i, unless the corresponding assumption was discharged. In that case, by theorem 7.1.1, either (→I) the conclusion of Π_i has higher degree, or (∨E) it is a negative subsentence of some assumption in Π_i and, consequently, a negative conclusion component of Π_i. Now consider the conclusion of Π_j. Take a parent track τ in Π_i. Suppose that τ is open and A is its assumption. By theorem 7.1.2, it is a negative subsentence of A and, by definition 7.1.5, also a distinctive conclusion component of Π_i. Finally, suppose that τ is closed and its leaf A was discharged. Then, by theorem 7.1.1, the conclusion of Π_i has higher degree than the conclusion of Π_j. □

Definition 7.1.7. Let Δ_Γ (resp. Δ_G) stand for the collection of distinctive assumption (resp. conclusion) components of a normal derivation from Γ to G. Consider sequences of subderivations $\langle \Gamma_0, G_0 \rangle \cdots \langle \Gamma_n, G_n \rangle$ where each element stands for a child subderivation of the previous one ($\langle \Gamma_0, G_0 \rangle$ being the main derivation). The derivations in the sequence are measured for their *complexity* as follows:

- if $\Delta_{G_j} < \Delta_{G_i}$ then $\langle \Gamma_j, G_j \rangle \prec \langle \Gamma_i, G_i \rangle$
- if $\Delta_{G_j} = \Delta_{G_i}$ then
  - if $\Delta_{\Gamma_j} < \Delta_{\Gamma_i}$ then $\langle \Gamma_j, G_j \rangle \prec \langle \Gamma_i, G_i \rangle$
  - if $\Delta_{\Gamma_j} \geq \Delta_{\Gamma_i}$ then $\langle \Gamma_j, G_j \rangle \succeq \langle \Gamma_i, G_i \rangle$

Theorem 7.1.4. *For any derivable argument, there is a normal derivation where the complexity decreases with the order of subderivations.*

Proof. I describe a method to shorten normal derivations by removing redundancies (loops). Consider a normal derivation where a subderivation Π_j has equal or higher complexity than its parent subderivation Π_i. By lemma 7.1.3, a parent track is open. Let A_i be its assumption. Now, there is also an open track in Π_j where another occurrence A_j of the same sentence is an analytic assumption, because otherwise Π_j would already have lower complexity than Π_i. Let Π_k be the descendant subderivation for the minor premiss of A_j. Replace Π_j with Π_k. Because Π_j has at least the same complexity as Π_i, any negative assumption components of Π_j is also a negative assumption component of Π_i. Therefore, any discharge of assumptions of Π_k can be transferred from Π_j to Π_i. This shortening process can be iterated until the resulting child of Π_i has lower complexity. $\square$

Example 7.1.8. Consider the derivation

$$\dfrac{\dfrac{\neg(\neg\neg A \to A)^{(1)} \quad \dfrac{\dfrac{\dfrac{\neg\neg A^{(2)} \quad \dfrac{\dfrac{\neg(\neg\neg A \to A)^{(3)} \quad \dfrac{\dfrac{\dfrac{\neg\neg A^{(5)} \quad \dfrac{\dfrac{\neg(\neg\neg A \to A)^{(6)} \quad \dfrac{A^{(7)}}{\neg\neg A \to A}}{\bot}}{\neg A}{\scriptstyle (7)}}{\bot}}{A}}{\neg\neg A \to A}{\scriptstyle (5)}}{\bot}}{\neg A}}{\bot}}{A}}{\neg\neg A \to A}{\scriptstyle (2)}}{\bot}}{\neg\neg(\neg\neg A \to A)}{\scriptstyle (1),(3),(6)}$$

Climbing up from the conclusion, the subderivation Π_1 for the second occurrence of $\neg\neg A \to A$ is more complex than its parent subderivation Π_2 for $\neg A$. More precisely, their distinctive conclusion components have equal degree and the distinctive assumption components of Π_2 have higher degree than the distinctive assumptions components of Π_1. In particular, $\neg\neg A$ is a distinctive assumption component of Π_2 but not of Π_1. As per theorem 7.1.4, the derivation can be shortened by replacing Π_2 with its descendent subderivation Π_3 for the third occurrence of $\neg\neg A \to A$.

$$\dfrac{\dfrac{\neg(\neg\neg A \to A)^{(1)} \quad \dfrac{\dfrac{\dfrac{\neg\neg A^{(2)} \quad \dfrac{\dfrac{\neg(\neg\neg A \to A)^{(3)} \quad \dfrac{A^{(4)}}{\neg\neg A \to A}}{\bot}}{\neg A}{\scriptstyle (4)}}{\bot}}{A}}{\neg\neg A \to A}{\scriptstyle (2)}}{\bot}}{\neg\neg(\neg\neg A \to A)}{\scriptstyle (1),(3)}$$

The discharge number 7 in Π_3 becomes discharge number 4 and discharges number 5 and 6 in Π_2 are discarded.

7.2 Validity

From a pragmatist perspective, a canonical argument start from open (usually complex) assumptions and, through applications of elimination rules, arrive at an atomic conclusion. Dummett also admits *basic rules* (or *boundary rules*) to determine deducibility among atomic sentences. In canonical arguments, these basic rules can be applied to an atomic conclusion in order to obtain further atomic consequences. However, since my main concern is with logical validity, I leave basic rules out of the picture and adapt Dummett's definitions accordingly, for the sake of simplicity.[3] I also adapt the definitions to the propositional case.

Definition 7.2.1. A sentence occurrence in an argument is *principal* if every sentence (inclusive) in the path down to the conclusion (exclusive) is a major premiss of an elimination rule.[4]

Definition 7.2.2. An argument is *proper* if at least one of its assumptions is principal.

The concept of proper argument is an essential component in the pragmatist notion of validity because proper arguments are built from the principal assumption by application of elimination rules. Arguments that do not follow this pattern are *improper*. The path from the principal assumption to the conclusion is the *principal path*.

The notion of canonical argument to be introduced later (definition 7.2.4) is based primarily on the notion of a proper argument. Even in proper arguments, however, the subargument for minor premisses of elimination rules may depend on auxiliary assumptions that arrive at the conclusion through improper means, i. e., through a path that is not solely composed of major premisses of elimination. These kind of improper subarguments for minor premisses of elimination are called *critical subarguments* (definition 7.2.5).

7.2.1 Canonical arguments and critical subarguments

The following definitions are adapted from Dummett's original definitions as explained in section 7.3. The core ideas, however, are preserved.

Definition 7.2.3. A sentence occurrence is *placid* if no sentence down the path to the conclusion is a horizontal minor premiss.

3. Dummett (1991, p. 273) explicitly stated the irrelevance of basic rules, or boundary rules, as he called them, to logical validity. Furthermore, by reflection on the pragmatist definition of validity (definition 7.2.8), it is easy to see that any basic rule in the complementation can be transferred to the valid canonical argument required, thus making no difference to which complex arguments are actually validated.

4. As a limiting case, in an argument consisting of a single occurrence of a sentence A, acting both as assumption and conclusion, A *is* principal, since the empty path from A to A satisfies the definition.

Definition 7.2.4. A *canonical* argument has the following properties:

(i) it is proper;

(ii) the subargument for any placid vertical minor premiss of an elimination rule is proper.

Definition 7.2.5. A *critical subargument* of a canonical argument is a non-canonical subargument whose conclusion is a horizontal minor premiss of an elimination rule.

In subarguments for minor premisses, the notion of canonicity deals with vertical and horizontal minor premisses differently. The subarguments for placid *vertical* minor premisses are considered independent auxiliary arguments and are thus required to be proper themselves. Ideally, the subarguments for *horizontal* minor premisses would also be proper (and canonical). However, *in general*, it is not possible to place any restrictions on the form of the subarguments for horizontal minor premisses: when not already canonical, those subarguments are critical, which means that the validity of the whole canonical argument would depend on their validity (definition 7.2.6).

Remark 7.2.1. The inference steps in canonical arguments consist primarily of applications of elimination rules: there is a principal path (which is composed of eliminations) and subarguments for minor premisses which are themselves either proper (again with a principal path composed of eliminations) or critical. Thus, a canonical argument has the general form

$$\dfrac{\dfrac{\dfrac{\cdots}{\cdots} \quad \dfrac{\dfrac{\cdots \quad \triangledown}{\cdots} \quad \triangledown}{\cdots} \quad \dfrac{\cdots \quad \triangledown}{\cdots}}{\cdots} \quad \triangledown}{\cdots}$$

where the inference steps are applications of elimination rules except for the critical subarguments (represented with "$\triangledown$" above), because the definitions impose no restrictions on their inference steps. If we were to ignore the critical subarguments, what remained could be called the *proper fraction* of the canonical argument and the corresponding sentence occurrences be called *proper occurrences*. In the proper fraction, in addition to the principal assumption, all other assumptions are principal assumptions of proper subarguments. In the context of canonical arguments, they are called, collectively, *proper leaves*, or *proper assumptions*, when undischarged throughout the argument.

Lemma 7.2.2. *The conclusion of a canonical argument is always a subsentence of some assumption, provided there is no proper occurrence of* $\bot$.

Proof. Let Π be a canonical argument with no proper occurrence of $\bot$. By definition 7.2.4, Π is proper and, by definition 7.2.2 and definition 7.2.1, it has a principal path of major premisses of applications of elimination rules from an assumption to the conclusion. In the principal path, the consequences of applications of $\wedge$E and $\rightarrow$E are subsentences of their respective major premisses. The interesting cases are applications of vertical rules ($\vee$E) since they figure a consequence which is not required to be a subsentence of the major premiss. By definition 7.2.4, the subarguments for minor premisses of vertical rules are proper. By definition 7.2.2 and definition 7.2.1, each vertical subargument has a path of eliminations from a proper leaf to the conclusion of the subargument. Now, if the proper leaf of a subargument for a vertical minor premiss was discharged by the corresponding application of $\vee$E, then the conclusion is, by induction hypothesis, a subsentence of its major premiss and hence a subsentence of the principal assumption. Otherwise, if the proper leaf was not discharged, then it is actually a proper assumption of the canonical argument and the conclusion is, by induction hypothesis, a subsentence of this assumption. □

Remark 7.2.3. It would perhaps be useful to make a parallel between the definitions above and concepts familiar from normalisation for intuitionistic natural deduction (section 7.1.2). For instance, notice that clause (ii) of definition 7.2.4 ensures that the segments in main tracks of canonical arguments are all major premisses of applications of elimination rules (except the last one). In main tracks of canonical arguments, the major premisses of vertical rules are followed by a corresponding discharged assumption, which (if not the last segment in the track) is also a major premiss of an elimination rule. As a result, the first sentence in a main track is a proper assumption and the last sentence (the conclusion of the canonical argument) is a subsentence of this assumption, provided $\bot$ does not occur in the main track. The first sentence in a main track, however, can be distinct from the principal assumption of the canonical argument, because assumptions discharged by vertical rules may not be principal in their respective proper minor subarguments. The tracks to whom the principal assumption belongs may be called *principal tracks.* By theorem 7.1.1, in normal derivations with empty synthetic parts, the main tracks are all principal tracks and the derivations are, therefore, canonical arguments. Critical subarguments are, inevitably, subarguments of higher order (clause (ii) of definition 7.1.1) although not every subargument of higher order must be critical, since it can be itself canonical (that is, in terms of derivations, when the synthetic parts of main tracks are empty).

Definition 7.2.6. A canonical argument is *valid* if all its critical subarguments are valid and of lower complexity.[5]

Definition 7.2.7. A *complementation* of an argument $\langle \Gamma, G \rangle$ is the result of replacing G by a valid canonical argument with the following properties:[6]

(i) it has G as principal assumption

(ii) it has an atomic conclusion

(iii) it has at most the same complexity[7]

Definition 7.2.8. An argument is *valid* if there is an effective method to transform any complementation of it into a valid canonical argument for the same conclusion from, at most, the same assumptions.

Definitions 7.2.6 to 7.2.8 should always be considered together since they define notions in terms of each other. For instance, the notion of valid canonical argument in definition 7.2.6 is defined in terms of the notion of valid argument which is itself only defined in definition 7.2.8.

The process of complementation consists basically in the application of elimination rules to the conclusion of the argument until we reach an atomic sentence. During complementation, the application of elimination rules figuring minor premisses can introduce auxiliary assumptions. Thus, the valid canonical argument required by definition 7.2.8 can depend on additional assumptions introduced by complementation.

Theorem 7.2.4 (Completeness of Intuitionistic Logic)**.** *If an argument $\langle \Gamma, G \rangle$ is valid, then there is a natural deduction derivation of G from Γ in intuitionistic logic.*

Proof. Suppose $\langle \Gamma, G \rangle$ is valid. By definition 7.2.8, for any complementation Π_c, we have a valid canonical argument Π_v for the same conclusion from, at most, the same assumptions. By definition 7.2.7, the complementations are obtained by replacing G with a valid canonical argument that has G as principal assumption. By definition 7.2.2, there is a principal path in Π_c from G to the

5. The complexity measure intended here is described in section 7.1.3, definition 7.1.7. The rationale behind it can be found in section 7.3.2.

6. Dummett's original definition has a special clause for when the conclusion G is an atomic sentence. His clause is subsumed here by canonical arguments consisting of a single occurrence of G (definition 7.2.1).

7. This ensures that complementations do not increase the complexity of the original argument.

atomic conclusion C which consists solely of applications of elimination rules. Furthermore, there can be auxiliary assumptions Δ in the subarguments for minor premisses of elimination rules in the principal path.

Complementation	Valid Canonical Argument	Derivation
$\begin{array}{c} \dfrac{\Gamma}{G}, \Delta \\ \vdots \\ C \quad (\Pi_c) \end{array}$	$\begin{array}{c} \Gamma, \Delta \\ \vdots \\ C \quad (\Pi_v) \end{array}$	$\begin{array}{c} \Gamma, \Delta \\ \vdots \\ C \quad (\Pi_d) \end{array}$

By definition 7.2.8, the valid canonical argument Π_v has C as conclusion and, at most, Γ, Δ as assumptions. We use Π_v as base and assume that we already have a natural deduction derivation Π_d for C from Γ, Δ obtained by recursive application of the procedure described here to the critical subarguments of Π_v.[8] Then, we go through the applications of elimination rules in the principal path and construct, through a process of inversion, a natural deduction derivation of G from Γ alone. Starting with C, we obtain a natural deduction derivation for each principal sentence in the principal path until we reach G (at which point we would have either discarded or discharged the auxiliary assumptions Δ). Since definition 7.2.8 ensures a valid canonical argument Π_v for *any* complementation Π_c, we are free to consider those complementations that are more convenient for the construction of our natural deduction derivation. We proceed case by case, where each case shows the derivation of the major premiss on the basis of derivations of the principal sentences that came before (ordered from C to G). For the cases of $\vee$E and $\rightarrow$E, which introduce auxiliary assumptions, I show how these assumptions can be either discarded ($\vee$E) or discharged ($\rightarrow$E). That is, for each occurrence of $\vee$E and $\rightarrow$E in the principal path of Π_c, I show how to obtain a derivation from only Γ, Δ_*, where Δ_* stands for the auxiliary assumptions except those assumptions that are being introduced at that particular inference. As a result, after going through all sentences in the principal path, we obtain a derivation of G which depends solely on Γ.

8. Recall that a canonical argument is mostly already a natural deduction derivation, except for the critical subarguments (remark 7.2.1). The termination of the recursive application of the procedure to the critical subarguments of Π_v is warranted by the complexity restrictions on critical subarguments (definition 7.2.6). Example 7.4.2 illustrates the recursive nature of the definitions. In the limiting case where assumptions Γ and conclusion G are atomic sentences, the argument is valid if, and only if, G is among the assumptions Γ.

($\wedge$E) Consider complementations by both elimination rules for conjunction

$$\dfrac{\begin{array}{c}\dfrac{\Gamma}{G,\Delta}\\ \vdots\\ A\wedge B\end{array}}{A}\ (\Pi_{c_1}) \qquad \dfrac{\begin{array}{c}\dfrac{\Gamma}{G,\Delta}\\ \vdots\\ A\wedge B\end{array}}{B}\ (\Pi_{c_2}) \qquad \dfrac{\begin{array}{c}\Gamma,\Delta\\ \vdots\\ A\end{array}\ (\Pi_{d_1}) \qquad \begin{array}{c}\Gamma,\Delta\\ \vdots\\ B\end{array}\ (\Pi_{d_2})}{A\wedge B}$$

From the derivations of A and B, the conjunction $A \wedge B$ is derived by $\wedge$I.

($\to$E) Consider a complementation where the minor premiss A is assumed.

$$\dfrac{\begin{array}{c}\dfrac{\Gamma}{G,\Delta_*}\\ \vdots\\ A\to B\end{array} \quad A}{B}\ (\Pi_c) \qquad\qquad \dfrac{\begin{array}{c}\Gamma,\Delta_*,[A]\\ \vdots\\ B\end{array}\ (\Pi_d)}{A\to B}$$

From a derivation of B from Γ, Δ, we apply $\to$I to obtain $A \to B$, discharging A.

($\bot$E) Consider a complementation where C is an atomic sentence that does not occur as a subsentence in either Γ or Δ. As a result, $\bot$E is the last rule applied. I show that Π_d contains a derivation of $\bot$ that depends, at most, on Γ, Δ.

$$\dfrac{\begin{array}{c}\dfrac{\Gamma}{G,\Delta}\\ \vdots\\ \bot\end{array}}{C}\ (\Pi_c) \qquad\qquad \dfrac{\boxed{\begin{array}{c}\Gamma,\Delta\\ \vdots\\ \bot\end{array}}}{C}\ (\Pi_d)$$

By definition 7.2.2, Π_d has a principal path from one of the assumptions Γ, Δ to C. Since C is not a subsentence of the principal assumption, it could only have been obtained by either $\bot$E or, possibly, a sequence of one or more applications of $\vee$E.[9] In the first case, we already have a derivation of $\bot$ from Γ, Δ. In the second case, by definition 7.2.4, the subarguments for vertical premisses are proper and hence have a principal path to C. We climb our way up the vertical minor premisses until we reach applications of a reductive elimination rule. The reductive

9. The cases where there is an application of $\bot$E with a complex consequence containing C and then further eliminations arriving at C are easily subsumed under the case where the corresponding application of $\bot$E has C directly as a consequence.

elimination rule in question can only be $\bot$E (lemma 7.2.2). We permute these reductive applications down the sequence of vertical premisses and thus obtain a derivation of $\bot$ from, at most, Γ, Δ.

($\vee$E) Consider a complementation of the form below, where $A \to C$ and $B \to C$ are assumed and C is atomic and does not occur as a subsentence in either Γ, A, B or auxiliary assumptions Δ_* (where Δ_* does not contain $A \to C$ and $B \to C$).

$$\dfrac{\begin{array}{c}\dfrac{\Gamma}{G, \Delta_*}\\ \vdots \\ A \vee B\end{array} \quad \dfrac{A \to C \quad [A]}{C} \quad \dfrac{B \to C \quad [B]}{C}}{C} \qquad (\Pi_c)$$

In the derivation Π_d, the conclusion C could have been obtained:

(a) by $\bot$E.

$$\dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ \bot\end{array}}{C} \quad (\Pi_d) \qquad\qquad \dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ \bot\end{array}}{A \vee B}$$

It is easy to derive $A \vee B$ instead by the same rule.

(b) by $\to$E from either $A \to C$ or, respectively, $B \to C$ as principal assumption.

$$\dfrac{A/B \to C \quad \boxed{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ A/B\end{array}}}{C} \quad (\Pi_d) \qquad\qquad \dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ A/B\end{array}}{A \vee B}$$

In either case, we obtain a derivation of $A \vee B$ from the subderivation of the horizontal minor premiss by $\vee$I.

(c) by a sequence of one or more applications of $\vee$E.

$$\dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ \therefore\end{array} \quad \dfrac{A/B \to C \quad \begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ A/B\end{array}}{C} \quad \dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ \bot\end{array}}{C}}{C} \quad (\Pi_d) \qquad \dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ \therefore\end{array} \quad \dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ A/B\end{array}}{A \vee B} \quad \dfrac{\begin{array}{c}\Gamma, \Delta_* \\ \vdots \\ \bot\end{array}}{A \vee B}}{A \vee B}$$

We then consider main tracks in Π_d (remark 7.2.3) and replace each occurrence of C in the segment by $A \vee B$ in accordance with previous cases (a) and (b).

The resulting derivation of $A \vee B$ abstains from assumptions $A \to C$ and $B \to C$. Any doubts can be dispelled by putting the derivation into normal form (theorem 7.1.1). □

Theorem 7.2.5 (Soundness of Intuitionistic Logic)**.** *If there is a natural deduction derivation of G from Γ, then the argument $\langle \Gamma, G \rangle$ is valid.*

Proof. Let Π_d be a normal derivation of G from Γ. Now, suppose Π_c is a complementation of Π_d from Γ, Δ to conclusion C. I show how to obtain a valid canonical argument from Γ, Δ to C. By definition 7.2.2, G is the first of a (possibly empty) sequence of major premisses of applications of elimination rules and, by theorem 7.1.1, the last of a (possibly empty) sequence of immediate premisses of applications of introduction rules. By induction on the degree of G, we perform reductions (section 2.3) until we obtain a deduction Π_v of C from Γ, Δ. By remark 7.2.3 and induction hypotheses on its critical subarguments, Π_v is a valid canonical argument for C from Γ, Δ. □

7.3 Prawitz's objection

There are subtle issues involved in the treatment of critical subarguments and Dummett was not able to get his definitions completely right. In particular, problems emerge when we consider a counterexample pointed out by Prawitz (2007, endnote 15):

> "The main fault [with Dummett's definitions] is that in a complementation of an argument, the minor premise of an implication elimination is only assumed. By not considering complementations where the minor premise is the end of an arbitrary argument (which is not possible to do in Dummett's definition, proceeding as it does by induction over the degree of arguments), the notion of validity becomes too weak. In particular, it cannot be shown that inferences by *modus ponens* are valid in general, because given two valid arguments Π and Σ for $A \to B$ and A, respectively, there is no guarantee that the result Δ of combining them in a *modus ponens* to conclude B is valid. For an actual counterexample, we may let B be atomic, Π be simply $A \to B$, which is a valid argument for $A \to B$ from $A \to B$, and Σ to be a valid argument for a nonatomic A from some hypotheses of higher degree than that of $A \to B$. Then Δ is canonical argument and is its own complementation, but it is not valid (Σ being of the same degree as Δ, nor can one find another valid canonical argument for B from the same hypotheses."

In his reply to Prawitz, Dummett (2007) acknowledges the problem. There are actually two different issues brought to light by Prawitz's counterexample. In the remainder of this section, I discuss these issues and indicate thereby the adaptations that I incorporated into the original definitions in order to avoid them. The adaptations, although elaborate, are fully in agreement with Dummett's overall philosophical outlook, particularly with respect to the treatment of assumptions.

7.3.1 Canonical atomism

Dummett (1991, pp. 284,285) discussed an example closely related to Prawitz's counterexample.

$$\dfrac{A \to B \qquad \dfrac{(A \to B) \to ((C \to C) \to A) \qquad A \to B}{A}}{B} \tag{7.1}$$

Notice that Dummett's example is basically an instance of Prawitz's counterexample: both consist of an argument where major premiss $A \to B$ stands as an assumption, and where there is a subargument for minor premiss A from assumptions of higher complexity than $A \to B$. In Dummett's discussion, however, the minor premiss A is *atomic*. This contrasts with Prawitz's counterexample where A is *complex*. The difference is important because, according to Dummett's original definition (which is divided into clauses), a canonical argument, besides being proper (clause iii), must have an atomic conclusion (clause i). Thus, the first problem revealed by Prawitz's counterexample is that, for complex A, there would be, in general, no canonical way to obtain A.

However, there is no conceptually compelling reason why canonical arguments must have atomic conclusions. After all, we should be able to obtain also complex sentences in a canonical manner.[10] In order to avoid this objection, I removed the requirement of atomic conclusion from Dummett's original definition of canonical argument and adapted the definition of complementation accordingly (clause (iii) of definition 7.2.7).

7.3.2 Stringency of the complexity restriction

When discussing his example (7.1), Dummett was concerned about improper and, therefore, non-canonical subarguments for minor premisses: if these kind

10. I suspect that Dummett only imposed the requirement of atomic conclusion on canonical arguments in order to simplify the formulation of his definition of complementation which, in general, should require that the principal path be as long as possible in order to afford a complete analysis of the conclusion.

of subarguments could have higher complexity than the principal assumption ($A \to B$, in this example), the definition of validity would be in danger of circularity. Dummett then presents a transformation that puts the improper subargument into proper form.

$$\dfrac{A \to B \qquad \dfrac{\dfrac{(A \to B) \to ((C \to C) \to A) \quad A \to B}{(C \to C) \to A} \qquad \dfrac{[C]}{C \to C}}{A}}{B} \tag{7.2}$$

Both arguments depend on the same assumptions but, in contrast with the original example (7.1), the transformed argument (7.2) displays a proper subargument for the minor premiss A, since there is a principal path from $(A \to B) \to ((C \to C) \to A)$ to A.

Apparently relying on the strength of this particular transformation alone, Dummett then introduces a *narrow* notion of validity for canonical arguments which restricts improper arguments for minor premisses to those of *strictly lower* degree, where the degree of an argument is the highest among the degrees of its assumptions and conclusion.

Although the transformation worked for that particular example, it is inadequate in general, at least if Dummett's notion of degree of an argument is used as complexity measure. Consider, for instance, the following proper argument:

$$\dfrac{\neg(A \vee \neg A) \qquad \dfrac{\neg(A \vee \neg A)}{A \vee \neg A}}{\bot}$$

The degree of the minor subargument is *equal* to the degree of the principal assumption $\neg(A \vee \neg A)$. In fact, $\neg(A \vee \neg A)$ occurs again as an assumption in the minor subargument. The fact that the minor subargument cannot be put into a proper form becomes clear when we replace it by its normal derivation in intuitionistic logic:

$$\dfrac{\neg(A \vee \neg A) \qquad \dfrac{\dfrac{\dfrac{\neg(A \vee \neg A) \quad \dfrac{A^{(1)}}{A \vee \neg A}}{\bot}}{\neg A}\,(1)}{A \vee \neg A}}{\bot}$$

Dummett's complexity restriction, as originally formulated, is therefore too stringent. The approach suggested by Prawitz (2007) and Schroeder-Heister

(2015) avoids this problem by dealing primarily with closed proofs, where the conclusion provides the adequate complexity measure, since there are no undischarged assumptions. Their approach thus differs unequivocally from Dummett's, especially with respect to the treatment of assumptions. I maintained Dummett's core approach through the adoption of an adequate complexity measure (definition 7.1.7), one that preserves soundness (theorem 7.1.4), instead of his original notion of degree of an argument.

7.4 A decision procedure

In order to illustrate the definitions and give some intuition about the construction described in the proof of theorem 7.2.4, it can be useful to work though some examples. The examples are meant to be an overall intuitive illustration of how the proof-theoretic definitions evaluate the validity of arguments. They are presented in the framework of a decision procedure that can be read off from the definitions.

The idea behind definition 7.2.8 is roughly that an argument is valid if, whatever we can obtain canonically from the conclusion, could as well be obtained from the assumptions. A procedure to evaluate validity can therefore be divided into two parts:

The `complementation` process determines what can be obtained from the conclusion.

The `search` process looks for a way to obtain the same thing from the assumptions.

Both `complementation` and `search` can employ only elimination rules — there are no introduction rules available. In line with definition 7.2.2, they are based on a similar method (let us call it *analysis*) of applying elimination rules to a sentence, taken as major premiss, until an atomic sentence is obtained (clause (ii) of definition 7.2.7). Thus, in the `complementation` process, the conclusion of the argument is analysed in order to see what atomic conclusions can be obtained (possibly under some additional auxiliary assumptions). In the `search` process, the assumptions are then analysed (one by one) in order to evaluate whether the same atomic conclusions can be obtained.

In the `complementation` process, the following simplifications are adopted, without loss of generality, with respect to $\vee$E and $\to$E (in agreement with the corresponding cases in the proof of theorem 7.2.4):

(→E) the minor premiss is assumed.

$$\frac{A \to B \quad A}{B}$$

Here, A is an additional assumption and will be available to `search`.

(∨E) applications are "flattened" with the help of implication.

$$\frac{A \vee B \qquad \dfrac{A \to C \quad [A]}{C} \qquad \dfrac{B \to C \quad [B]}{C}}{C}$$

In order to maintain generality, C stands for a sentence that does not occur as a subsentence either in the assumptions or the conclusion. Here, $A \to C$ and $B \to C$ are assumed and will be available to `search`.

(⊥E) applications are abstained. The `search` will then target ⊥. Notice that these simplifications are limited to the `complementation` process and do *not* carry over to the `search` process where, naturally, applications of ⊥E are not abstained.

Example 7.4.1. A definition of validity is expected to provide precise criteria for the validity of arguments and, for the pragmatist definitions in particular, these criteria are supposed to resort to elimination rules only (without assistance from introduction rules). Consider a simple, but not trivial, argument

$$\frac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}$$

and let us evaluate its validity with respect to the pragmatist definitions.

First, we investigate what can be obtained canonically from the conclusion by means of `complementation`:

$$\frac{\dfrac{\dfrac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}}{A \to B} \qquad A}{B} \qquad\qquad \frac{\dfrac{\dfrac{A \to (B \wedge C)}{(A \to B) \wedge (A \to C)}}{A \to C} \qquad A}{C}$$

There are two complementations, with conclusions B and C, respectively, and the assumptions $A \to (B \wedge C)$ and A. In order to establish validity, we must now find canonical arguments from $A \to (B \wedge C)$ and A to B, and from $A \to$

$(B \wedge C)$ and A to C.[11] The `search` for these canonical arguments can be done mechanically by analysing the assumptions one by one, where some heuristics could be employed to sort out the most promising candidates. In this example, we have few assumptions and don't need much heuristics to see that $A \to (B \wedge C)$ is the best candidate:

$$\dfrac{\dfrac{A \to (B \wedge C) \quad A}{B \wedge C}}{B} \qquad\qquad \dfrac{\dfrac{A \to (B \wedge C) \quad A}{B \wedge C}}{C}$$

The procedure is thus revealed to be strong enough to validate, not only the introduction rules on the basis of the elimination rules, but also complex arguments whose derivation would require both eliminations *and* introduction rules.

Example 7.4.2. Regarded as a decision algorithm, the procedure for evaluation of validity based on elimination rules is not so straightforward and uncomplicated as example 7.4.1 makes it out to be. In the general case, the procedure may involve recursion and backtracking. The `search` process can deliver candidates with critical subarguments, which would demand a recursive call to evaluate their legitimacy (definition 7.2.6). If unsuccessful, the process backtracks and tries out the analysis on a different assumption. Consider, for instance, the argument

$$\dfrac{A \to \neg\neg B}{\neg\neg(\neg A \vee B)}$$

The `complementation` below stops at the conclusion $\bot$, before an application of $\bot$E, in accordance with the aforementioned simplifications to the complementation process.

$$\dfrac{\dfrac{A \to \neg\neg B}{\neg\neg(\neg A \vee B)} \quad \neg(\neg A \vee B)}{\bot} \tag{C1}$$

The `search` process has assumptions $A \to \neg\neg B$ and $\neg(\neg A \vee B)$ to try out in order to obtain $\bot$. For simplicity of exposition, let us heuristically select $\neg(\neg A \vee B)$, although we could as well have first unsuccessfully tried $A \to \neg\neg B$

11. The assumptions of the complementations happen to be the same in this example. In the general case, however, they have to be considered separately, e.g. each complementation has their own assumptions and conclusion. In order to establish validity, we must then show that the conclusion of the complementation can be obtained from the assumptions of the complementation, *for every complementation.*

out and then backtracked here.

$$\frac{\neg(\neg A \lor B) \quad \neg A \lor B}{\bot}$$

Now, notice that $\neg A \lor B$ itself is not available among our assumptions. Therefore we *presume* that $\neg A \lor B$ can, in fact, be obtained from the assumptions that *are* available to us, and recall the procedure recursively on the critical subargument enclosed in a box below.[12]

$$\frac{\neg(\neg A \lor B) \quad \boxed{\dfrac{A \to \neg\neg B \quad \neg(\neg A \lor B)}{\neg A \lor B}}}{\bot} \tag{S1}$$

In the `complementation` of our recursive call, we again adhere to aforementioned simplifications and use C as the conclusion of $\lor$E since it does not occur anywhere else.

$$\frac{\dfrac{A \to \neg\neg B \quad \neg(\neg A \lor B)}{\neg A \lor B} \quad \dfrac{\neg A \to C \quad [\neg A]}{C} \quad \dfrac{B \to C \quad [B]}{C}}{C} \tag{C2}$$

In order to obtain the foreign C, the `search` must either (1) obtain $\bot$, and thereby C, or (2) obtain one of the disjuncts and thereby obtain C from the corresponding assumption, either $\neg A \to C$ or $B \to C$, or yet (3) obtain C by $\lor$E from a disjunctive principal sentence, whereby we may use the disjuncts as additional assumptions on the `search` for proper subarguments for the respective vertical minor premisses. We examine the second option and choose assumption $\neg A \to C$. The other one may be discarded.

$$\frac{\neg A \to C \quad \boxed{\dfrac{A \to \neg\neg B \quad \neg(\neg A \lor B)}{\neg A}}}{C} \tag{S2}$$

The next recursive step reveals an important aspect of the definitions. Consider the `complementation`.

$$\frac{\dfrac{A \to \neg\neg B \quad \neg(\neg A \lor B)}{\neg A} \quad A}{\bot} \tag{C3}$$

12. Notice that $\neg(\neg A \lor B)$ appears twice: as major premiss and also as an assumption of the critical subargument. This cannot be avoided in general and is related to the problem with contraction in the search for proofs in the sequent calculus (Došen 1987; Dyckhoff 1992; Hudelmaier 1993).

In the candidate (S3) below, if we were to retain all the assumptions available for the next recursive call, that is, if $A \to \neg\neg B$ and A where *both* passed as assumptions to the critical subargument enclosed in a box, we would be in danger of running into a vicious circle (a loop): after the `complementation` (C4) below, the candidate (S2) above may need to be considered again by the `search`. Indeed, by definition 7.1.7, the argument from $\{A \to \neg\neg B, A, \neg(\neg A \vee B)\}$ to $\neg B$ has higher complexity than the critical subargument in (S2), because its conclusion $\neg B$ has the same degree than $\neg A$ and it has A as an additional assumption. Therefore, for the particular case with $A \to \neg\neg B$ as principal assumption, the `search` must consider only candidates where A or some other assumptions are left out of the critical subargument, on pain of violating the complexity restriction. As it turns out, we do not need $A \to \neg\neg B$ either.

$$\frac{\dfrac{A \to \neg\neg B \quad A}{\neg\neg B} \qquad \boxed{\dfrac{\neg(\neg A \vee B)}{\neg B}}}{\bot} \tag{S3}$$

More recursion.

$$\frac{\dfrac{\neg(\neg A \vee B)}{\neg B} \qquad B}{\bot} \tag{C4}$$

We see the complexity restriction at work again in the candidate offered by the `search` below (notice that $\neg(\neg A \vee B)$ is left out of the critical subargument).

$$\frac{\neg(\neg A \vee B) \qquad \boxed{\dfrac{B}{\neg A \vee B}}}{\bot} \tag{S4}$$

I think that the procedure should be clear enough by now for us to omit the last recursive call.

The construction described in theorem 7.2.4 can be applied to the canonical arguments produced by `complementation` and `search` in order to obtain a derivation.

$$\dfrac{\dfrac{\neg(\neg A \vee B)^{(1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \to \neg\neg B \quad A^{(2)}}{\neg\neg B} \qquad \dfrac{\dfrac{\neg(\neg A \vee B)^{(3)} \quad \dfrac{B^{(4)}}{\neg A \vee B}}{\bot}}{\neg B}\,(4)}{\bot}}{\neg A}\,(2)}{\neg A \vee B}}{\bot}}{\neg\neg(\neg A \vee B)}\,(1),(3)$$

The derivation contains four tracks. If we order the tracks from one to four and divide them into their analytic and synthetic parts, they correspond roughly to the complementation and search processes of the procedure: (C1), [(C2), (C3)], (C4) and (C5) (omitted) correspond to the synthetic parts of tracks 1, 2, 3 and 4; (S1), (S3) and (S4) correspond to the analytic parts of tracks 1, 2 and 3 (the analytic part of track 4 is empty). The simplifications adopted with respect to $\vee$E in the `complementation` process resulted in a dedicated recursive step for applications of $\vee$I in the derivation (in track 2, (C2) and (S2); in track 4, (C5) and (S5)). This seems a reasonable exchange against the achieved separation between the processes and deterministic character of the `complementation`.

8 Discussion and Prospects

Dummett's approach to proof-theoretic validity differs substantially from standard approaches. The most noteworthy difference is rejection of the placeholder view of assumptions. As a result, conceptual and technical advantages are gained for the notion of validity based on introduction rules and complete adequacy to intuitionistic logic is achieved for the notion of validity based on the elimination rules.

Standard definitions of validity have often been biased towards introduction rules and verificationism. Comparatively, notions of validity based on the elimination rules has received far less attention (Dummett 1991; Prawitz 2007; Schroeder-Heister 2015). I hope the results in chapter 7 would help to tip the scale. In addition to the technical results, I discuss some philosophical and conceptual reasons to prefer the pragmatist approach.

8.1 Schematicity and independence

In contrast with the verificationist notion, the pragmatist notion of validity is, by design, completely schematic. In supplementations, the situation could change drastically if an atomic sentence needed to be replaced throughout by a complex sentence (section 6.5.1). In complementations, on the other hand, this kind of replacement would not disturb the validation procedure in the slightest.

In the verificationist scenario, verifications for atomic sentences must be provided in the form of atomic bases and they are an indispensable component in the definition of validity. As a result, the verificationist notion of validity becomes highly sensitive to changes in the basic framework. From the point of view of logical validity, there is no reason why bases should matter at all. In this respect, the pragmatist notion of validity offers a better approach because it is fundamentally independent from bases, although it could incorporate them without any problems.

8.2 Invalidity, Kripke semantics and counterexamples

The purpose of a logical semantics is not only to ascertain validity but also to provide a criterion for invalidity. The question then could be raised: How does a proof-theoretic semantics establish invalidity, since it cannot avail itself of countermodels and, consequently, the usual notion of counterexample?

According to definition 7.2.8, a counterexample is a complementation such that no valid canonical argument can be provided for the same conclusion from, at most, the same assumptions. Once the simplifications described in section 7.4 are adopted, complementations become deterministic. The valid or invalid verdict would then rest upon the canonical analysis of the available assumptions: if they do not afford the desired conclusion, then the argument is invalid. In the worst case scenario, after the necessary recursive steps over critical subarguments, the verdict would ultimately rest upon the validity of simple atomic arguments.

For comparison, consider Kripke semantics for intuitionistic logic (Troelstra and van Dalen 1988, § 2.5). If there is a valid canonical argument for an atomic sentence c from assumptions Γ, then c is forced by any node κ in a Kripke model $\mathcal{K}$ such that, for each assumption A in Γ, A is valid in $\mathcal{K}$. Therefore, if the assumptions in an argument are associated with nodes at Kripke models in which they are valid, then pragmatist complementation counterexamples correspond to invalidating nodes in Kripke countermodels.

Example 8.2.1. I show that Peirce's rule

$$\frac{(A \to B) \to A}{A}$$

is invalid. By definition 7.2.8, there is a complementation that cannot be transformed into a valid canonical argument for the same conclusion from, at most, the same assumptions. Suppose, for simplicity, that A and B are distinct atomic sentences. Then, Peirce's rule is, so to say, its own complementation (definition 7.2.7). A valid canonical argument for A from $(A \to B) \to A$

$$\frac{(A \to B) \to A \quad A \to B}{A}$$

would need $A \to B$ as an additional assumption unless a critical subargument for $A \to B$ could be validly obtained from $(A \to B) \to A$ (definition 7.2.6).

$$\frac{(A \to B) \to A \qquad \boxed{\dfrac{(A \to B) \to A}{A \to B}}}{A}$$

But this is not the case, because the complementation of the critical subargument

$$\dfrac{\dfrac{(A \to B) \to A}{A \to B} \qquad A}{B}$$

cannot be transformed into a valid canonical argument for B from $(A \to B) \to A$ and A since none of these assumptions can be principal in a canonical argument for B (definition 7.2.4).

Example 8.2.2. I show that the argument

$$\dfrac{A \to (B \lor C)}{(A \to B) \lor (A \to C)}$$

is invalid. For simplicity, assume that A, B, C and D are distinct atomic sentences and consider the complementation

$$\dfrac{\dfrac{A \to (B \lor C)}{(A \to B) \lor (A \to C)} \quad \dfrac{(A \to B) \to D \quad [A \to B]}{D} \quad \dfrac{(A \to C) \to D \quad [A \to C]}{D}}{D}$$

Any purpoted valid canonical argument for D must have $(A \to C) \to D$ and/or $(A \to B) \to D$ as proper assumptions (lemma 7.2.2). None of those candidates, however, are valid

$$\dfrac{\dfrac{A \to (B \lor C) \quad \boxed{\dfrac{A \to (B \lor C)}{A}}}{B \lor C} \quad \dfrac{(A \to B) \to D \quad \boxed{\dfrac{A \to (B \lor C) \quad B}{A \to B}}}{D} \quad \dfrac{(A \to C) \to D \quad \boxed{\dfrac{A \to (B \lor C) \quad C}{A \to C}}}{D}}{D}$$

$$\dfrac{(A \to B) \to D \quad \boxed{\dfrac{A \to (B \lor C)}{A \to B}}}{D} \qquad \dfrac{(A \to C) \to D \quad \boxed{\dfrac{A \to (B \lor C)}{A \to C}}}{D}$$

because not all of their critical subarguments are valid (definition 7.2.6).

8.3 Towards proof-theoretic semantics for philosophical logics

As already remarked, the pragmatist notion of validity embraces assumptions and hypothetical arguments as first class citizens, so to speak. A direct benefit of this feature is that it opens up the possibility to handle substructural logics by refining the way assumptions are collected and managed. This is why I avoided talking about "sets" of assumptions but instead preferred the more ambiguous term "collection".

For example, the pragmatist procedure can handle relevant implications if some minor adaptations are made. With assumptions collected into sets, definition 7.2.8 could be amended to require valid canonical arguments to depend on *exactly the same set* of assumptions. Another substructural implication can be handled if assumptions are collected into multisets. In a similar manner, various restrictions to the structural rules of weakening and contraction could be imposed. In fact, I would expect that the core of the pragmatist notion is robust enough to account for many of the substructural systems described by Došen (1988). Further research on the topic must be conducted before this expectation can be either fulfilled or frustrated. The prospects, nevertheless, are very promising.

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