Proofs and Dialogues
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Parallel Dialogue Games and Hypersequents for Intermediate Logics

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Motivation:
Understanding Hypersequent Calculi

Avron's communication rule (for Gödel-Dummett logic G_∞):

$$\frac{\Pi_1, \Pi_2 \longrightarrow C_1 \mid \mathcal{H} \qquad \Lambda_1, \Lambda_2 \longrightarrow C_2 \mid \mathcal{H}}{\Pi_1, \Lambda_1 \longrightarrow C_1 \mid \Pi_2, \Lambda_2 \longrightarrow C_2 \mid \mathcal{H}} \; (com.)$$

'**Avron-Baaz-claim:**'

The communication rule models the exchange of information between parallel processes.

Consequently:

G_∞ bears the same relation to parallel programs as intuitionistic logic bears to sequential programs.

Dialogues as foundations

Imagine a dialogue, where a Proponent **P** tries to defend a logically complex statement against attacks by an Opponent **O**.

Central idea:

logical validity of F is identified with '**P** can always win the dialogue starting with her assertion of F'

Some basic features of Lorenzen style dialogues:

- attacking moves and corresponding defense moves refer to connectives (or quantifiers)
- both, **P** and **O**, may launch attacks and defend against attacks during the course of a dialogue
- moves alternate strictly between **P** and **O**

Logical dialogue rules:

X/**Y** stands for **P**/**O** or **O**/**P**

statement by **X**	attack by **Y**	defense by **X**
$A \wedge B$	l? or r? (**Y** chooses)	A or B, accordingly
$A \vee B$	?	A or B (**X** chooses)
$A \supset B$	A	B

Note: $\neg A$ abbreviates $A \supset \bot$.

Winning conditions for P:

W: **O** has already granted **P**'s current formula.

W$\bot$: **O** has granted $\bot$.

Structural rules:

Start:	**O** starts by attacking **P**'s initial assertion (formula).
Alternate:	Moves strictly alternate between **O** and **P**.
Atom:	Atomic formulas (including $\perp$) can neither be attacked nor defended by **P**.
'E-rule':	Each move of **O** reacts directly to the immediately preceding move by **P**.

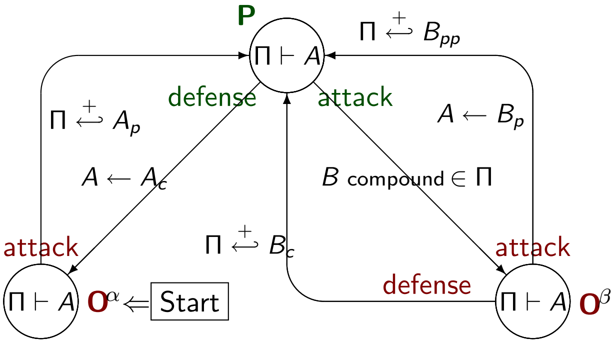
Winning strategies

Definition:
A **winning strategy** (for **P**) is a finite tree, whose branches are dialogues that end in winning states for **P**, s.t.
– **P**-nodes have (at most) one successor;
– **O**-nodes have successors for each possible next move by **O**.

Note:
Dialogues are traces of the corresponding state transition system.

Winning strategies arise by 'unwinding' the state transition system.

Dialogue as state transitions ($\supset$-fragment):



Adequacy of the dialogue game for I

Theorem (Lorenzen, Lorenz, Felscher, ...):
P has a winning strategy when initially asserting F

if and only if

F is valid according to intuionistic logic (**I**).

Version of the adequacy theorem needed here:

Theorem:

Winning strategies correspond to cut-free **LI**′-proofs.

Remark on adequacy proofs:
The correspondence between winning strategies and analytic proofs has been shown many times – also for variants adequate for **classical**, **modal**, (fragments of) **linear** and many **other logics**.
After Felscher: Barth, Krabbe, Keiff, Rahman, Blass, Sorensen and Urzyczyn(!), ...

LI′: the proof search friendly version of LI (LJ?)

Axioms:

'confine weakening to axioms':

$$\bot, \Pi \longrightarrow C \quad \text{and} \quad A, \Pi \longrightarrow A$$

Logical rules:

'keep a copy of the main (i.e. reduced) formula around':

$$\frac{A \supset B, \Pi \longrightarrow A \qquad B, A \supset B, \Pi \longrightarrow C}{A \supset B, \Pi \longrightarrow C}\ (\supset, l)$$

$$\frac{A, \Pi \longrightarrow B}{A, \Pi \longrightarrow A \supset B}\ (\supset, r)$$

HLI′: A hypersequent calculus for intuitionistic logic

Exactly as **LI′** except for the presence of side hypersequents:

Axioms:

$$\bot, \Pi \longrightarrow C \mid \mathcal{H} \quad \text{and} \quad A, \Pi \longrightarrow A \mid \mathcal{H}$$

Logical rules:

$$\frac{A \supset B, \Pi \longrightarrow A \mid \mathcal{H} \qquad B, A \supset B, \Pi \longrightarrow C \mid \mathcal{H}}{A \supset B, \Pi \longrightarrow C \mid \mathcal{H}} \ (\supset, l)$$

$$\frac{A, \Pi \longrightarrow B \mid \mathcal{H}}{A, \Pi \longrightarrow A \supset B \mid \mathcal{H}} \ (\supset, r)$$

Note:

The side hypersequents are clearly redundant here, but may be useful in representing choices in proof search (once the ‘obvious’ external structural rules are in place . . .)

Internal structural rules:

$$\frac{A, A, \Pi \longrightarrow C \mid \mathcal{H}}{A, \Pi \longrightarrow C \mid \mathcal{H}}\ (\textit{I-contr.}) \qquad \frac{\Pi \longrightarrow C \mid \mathcal{H}}{A, \Pi \longrightarrow C \mid \mathcal{H}}\ (\textit{I-weakening})$$

$$\frac{\Pi \longrightarrow A \mid \mathcal{H} \quad A, \Pi \longrightarrow C \mid \mathcal{H}'}{\Pi \longrightarrow C \mid \mathcal{H} \mid \mathcal{H}'}\ (\textit{cut})$$

Remember: cut and internal weakening are redundant!

External structural rules:

$$\frac{\mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}}\ (\textit{E-weakening}) \qquad \frac{\Pi \longrightarrow C \mid \Pi \longrightarrow C \mid \mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}}\ (\textit{E-contr.})$$

Note:

E-weakening records the dismissal of an alternative in proof search.
E-contraction records a 'backtracking point' for such an alternative.

Parallel dialogue games

General features of our form of parallelization:

- Ordinary dialogues (**I**-dialogues) appear as subcases of the more general parallel framework.
- **P** may initiate additional dialogues by 'cloning'.
- To win a set of parallel dialogues, **P** has to win at least one of the component **I**-dialogues.
- Synchronization between parallel **I**-dialogues is invoked by **P**'s decision to merge some **I**-dialogues ('component dialogues') into one. **O** may react to this in different ways.

Notions for parallel dialogue games

A parallel **I**-dialogue (P-**I-dialogue**) is a sequence of global states connected by internal or external moves.

Global state:
$\{\Pi_1 \vdash_{\iota 1} C_1,\ \ldots,\ \Pi_n \vdash_{\iota n} C_n\}$
(Set of uniquely indexed component **I**-dialogue sequents.)

Internal move:
Set of **I**-dialogue moves: at most one for each component.

External move:
May add or remove components, but does not change the status — **P**'s or **O**'s turn to move — of existing components.

Basic external moves:

fork: **P** duplicates a **P**-component of the current global state.

cancel: **P** removes an arbitrary **P**-component (if the global state contains another **P**-component).

Towards proving adequacy: Sequentialized and normal P-$\mathbf{I}$-dialogues

Sequentiality: internal moves are singletons.

Normality:

- **P**-moves are immediately followed by **O**-moves referring to the same component(s)
- external moves (possibly consisting of a **P**-**O**-round) are followed by **P**-moves

Lemma:

Every finite P-$\mathbf{I}$-dialogue can be translated into an equivalent sequentialized and normal P-$\mathbf{I}$-dialogue.

Theorem:

Winning strategies for sequentialized and normal P-$\mathbf{I}$-dialogues correspond to $\mathbf{HLI}'$-proofs.

Example: Characterizing Gödel-Dummett logic

HLC$'$ is obtained from **HLI**$'$ by adding:

$$\frac{\Pi_1, \Pi_2 \longrightarrow C_1 \mid \mathcal{H} \qquad \Pi_1, \Pi_2 \longrightarrow C_2 \mid \mathcal{H}}{\Pi_1 \longrightarrow C_1 \mid \Pi_2 \longrightarrow C_2 \mid \mathcal{H}} \ (com')$$

This correponds to the following 'synchronisation rule':

lc-merge:
1. **P** picks two **P**-components $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$.
2. **O** chooses either C_1 or C_2 as the current formula of the merged component with granted formulas $\Pi_1 \cup \Pi_2$.

Theorem:

Winning strategies for P-**I**-dialogues with lc-merge can be translated into cut-free **HLC**$'$-proofs, and vice versa.

Other forms of synchronization:

System	rule	external move(s)
P-**Cl**	class	**P** merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} C$ into $\Pi \cup \Gamma \vdash_{\iota 2} C$
P-**LQ**	lq	**P** merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} \bot$ into $\Pi \cup \Gamma \vdash_{\iota 2} \bot$
P-**LC**	lc	**P** picks $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$ **O** chooses $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$ or $\Pi_1 \cup \Pi_2 \vdash_{\iota 2} C_2$
P-**sLC**	lc0	**P** picks $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$ **O** chooses $\Pi_2 \vdash_{\iota 1} C_1$ or $\Pi_1 \vdash_{\iota 2} C_2$
	sp	**P** merges $\Pi \vdash_{\iota 1} C$ and $\Gamma \vdash_{\iota 2} C$ into $\Pi \cup \Gamma \vdash_{\iota 2} C$
P-$\mathbf{G_n}$	g_n	**P** picks the components $\Pi_1 \vdash_{\iota 1} C_1$, and $\ldots \Pi_{n-1} \vdash_{\iota[n-1]} C_{n-1}$, and $\Pi_n \vdash_{\iota n}$ **O** chooses one of $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$, $\Pi_2 \cup \Pi_3 \vdash_{\iota 2} C_2$, $\ldots$, or $\Pi_{n-1} \cup \Pi_n \vdash_{\iota[n-1]} C_{n-1}$

Concluding remarks

'Avron-Baaz-claim': We interpreted the communication rule in terms of 'joining resources' of parallel dialogue runs.

Models of proof search: **P**-**O** as 'Client-Server' view allows to model different proof search strategies, including distributed search.

Uniformity and flexibility: All 'analytic' intermediate logics — including intuitionistic and classical logic — can be characterized by the same basic game augmented by somewhat different forms of 'synchronisation'.

Beyond intermediate logics: Resource sensitivity and modalities can be handled elegantly in the dialogue format!
$\Longrightarrow$ Games for
Łukasiewicz logic(s),
contraction free intuitionistic logics,
Urquhart's 'basic logic',
...