Giles's Game and the Proof Theory of Łukasiewicz Logic

George Metcalfe

Mathematics Institute
University of Bern

Joint work with Christian G. Fermüller

Proof and Dialogues

27 February 2011, Tübingen

A First Dialogue

You (Chris)	**Me (George)**
There seems to be no "good" proof system for **Łukasiewicz logic**. . .	
	But I have just found an elegant **hypersequent calculus**!
Nice! But what do hypersequents *mean* in this system?	
	Well, I have a complicated translation into the logic. . .
Perhaps **dialogue games** provide an answer?	

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles's game**.	
	But can we make this *formal*?

A First Dialogue

You (Chris)

There seems to be no "good" proof system for **Łukasiewicz logic**...

Me (George)

But I have just found an elegant **hypersequent calculus**!

Nice! But what do hypersequents *mean* in this system?

Well, I have a complicated translation into the logic...

Perhaps **dialogue games** provide an answer?

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles's game**.

But can we make this *formal*?

A First Dialogue

You (Chris)	**Me (George)**
There seems to be no "good" proof system for **Łukasiewicz logic**. . .	
	But I have just found an elegant **hypersequent calculus**!
Nice! But what do hypersequents *mean* in this system?	
	Well, I have a complicated translation into the logic. . .
Perhaps **dialogue games** provide an answer?	

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles's game**.	
	But can we make this *formal*?

A First Dialogue

You (Chris)

There seems to be no “good” proof system for **Łukasiewicz logic**. . .

Nice! But what do hypersequents *mean* in this system?

Perhaps **dialogue games** provide an answer?

Me (George)

But I have just found an elegant **hypersequent calculus**!

Well, I have a complicated translation into the logic. . .

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles’s game**.

But can we make this *formal*?

A First Dialogue

You (Chris)	**Me (George)**
There seems to be no "good" proof system for **Łukasiewicz logic**. . .	
	But I have just found an elegant **hypersequent calculus**!
Nice! But what do hypersequents *mean* in this system?	
	Well, I have a complicated translation into the logic. . .
Perhaps **dialogue games** provide an answer?	

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles's game**.	
	But can we make this *formal*?

A First Dialogue

You (Chris)	**Me (George)**
There seems to be no “good” proof system for **Łukasiewicz logic**. . .	
	But I have just found an elegant **hypersequent calculus**!
Nice! But what do hypersequents *mean* in this system?	
	Well, I have a complicated translation into the logic. . .
Perhaps **dialogue games** provide an answer?	

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles’s game**.	
	But can we make this *formal*?

A First Dialogue

You (Chris)	**Me (George)**
There seems to be no "good" proof system for **Łukasiewicz logic**. . .	
	But I have just found an elegant **hypersequent calculus**!
Nice! But what do hypersequents *mean* in this system?	
	Well, I have a complicated translation into the logic. . .
Perhaps **dialogue games** provide an answer?	

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles's game**.	
	But can we make this *formal*?

A First Dialogue

You (Chris)	**Me (George)**
There seems to be no "good" proof system for **Łukasiewicz logic**. . .	
	But I have just found an elegant **hypersequent calculus**!
Nice! But what do hypersequents *mean* in this system?	
	Well, I have a complicated translation into the logic. . .
Perhaps **dialogue games** provide an answer?	

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles's game**.	
	But can we make this *formal*?

A First Dialogue

You (Chris)	**Me (George)**
There seems to be no “good” proof system for **Łukasiewicz logic**. . .	
	But I have just found an elegant **hypersequent calculus**!
Nice! But what do hypersequents *mean* in this system?	
	Well, I have a complicated translation into the logic. . .
Perhaps **dialogue games** provide an answer?	

(**Daniele**: Have you considered these papers by Robin Giles?)

Aha! Hypersequent proofs are *strategies* in **Giles’s game**.

But can we make this *formal*?

This Talk

I will address these questions:

- What is **Giles's game**?
- How does it relate to **Łukasiewicz logic**?
- How does it relate to the **proof theory** of Łukasiewicz logic?
- What more can be done with this approach?

G. Metcalfe and C. G. Fermüller. Giles's Game and the Proof Theory of Łukasiewicz Logic. *Studia Logica*, 92(1):27–61 (2009).

This Talk

I will address these questions:

- What is **Giles's game**?
- How does it relate to **Łukasiewicz logic**?
- How does it relate to the **proof theory** of Łukasiewicz logic?
- What more can be done with this approach?

G. Metcalfe and C. G. Fermüller. Giles's Game and the Proof Theory of Łukasiewicz Logic. *Studia Logica*, 92(1):27–61 (2009).

This Talk

I will address these questions:

- What is **Giles's game**?
- How does it relate to **Łukasiewicz logic**?
- How does it relate to the **proof theory** of Łukasiewicz logic?
- What more can be done with this approach?

G. Metcalfe and C. G. Fermüller. Giles's Game and the Proof Theory of Łukasiewicz Logic. *Studia Logica*, 92(1):27–61 (2009).

This Talk

I will address these questions:

- What is **Giles's game**?
- How does it relate to **Łukasiewicz logic**?
- How does it relate to the **proof theory** of Łukasiewicz logic?
- What more can be done with this approach?

G. Metcalfe and C. G. Fermüller. Giles's Game and the Proof Theory of Łukasiewicz Logic. *Studia Logica*, 92(1):27–61 (2009).

This Talk

I will address these questions:

- What is **Giles's game**?
- How does it relate to **Łukasiewicz logic**?
- How does it relate to the **proof theory** of Łukasiewicz logic?
- What more can be done with this approach?

G. Metcalfe and C. G. Fermüller. Giles's Game and the Proof Theory of Łukasiewicz Logic. *Studia Logica*, 92(1):27–61 (2009).

An Overview of the Game

In the 1970s, Robin Giles introduced a two-player dialogue game

You claim...	**I claim...**
$\varphi_1, \dots, \varphi_n$	$\psi_1, \dots, \psi_m$

consisting of two parts...

1. **Atomic statements** refer to experiments with a fixed probability of a positive result, and the players pay 1€ to their opponent for each incorrect statement – the winner *expects* not to lose money.
2. **Compound statements** are attacked or granted by the opposing player based on natural dialogue rules.

R. Giles. A non-classical logic for physics.
Studia Logica, 4(33):399–417 (1974).

R. Giles. Łukasiewicz logic and fuzzy set theory
International Journal of Man-Machine Studies, 8(3):313–327 (1976).

An Overview of the Game

In the 1970s, Robin Giles introduced a two-player dialogue game

You claim. . .	**I claim. . .**
$\varphi_1, \ldots, \varphi_n$	$\psi_1, \ldots, \psi_m$

consisting of two parts. . .

1. **Atomic statements** refer to experiments with a fixed probability of a positive result, and the players pay 1€ to their opponent for each incorrect statement – the winner *expects* not to lose money.
2. **Compound statements** are attacked or granted by the opposing player based on natural dialogue rules.

R. Giles. A non-classical logic for physics.
Studia Logica, 4(33):399–417 (1974).

R. Giles. Łukasiewicz logic and fuzzy set theory
International Journal of Man-Machine Studies, 8(3):313–327 (1976).

An Overview of the Game

In the 1970s, Robin Giles introduced a two-player dialogue game

You claim...	**I claim...**
$\varphi_1, \dots, \varphi_n$	$\psi_1, \dots, \psi_m$

consisting of two parts...

1. **Atomic statements** refer to experiments with a fixed probability of a positive result, and the players pay 1€ to their opponent for each incorrect statement – the winner *expects* not to lose money.
2. **Compound statements** are attacked or granted by the opposing player based on natural dialogue rules.

R. Giles. A non-classical logic for physics.
Studia Logica, 4(33):399–417 (1974).

R. Giles. Łukasiewicz logic and fuzzy set theory
International Journal of Man-Machine Studies, 8(3):313–327 (1976).

An Overview of the Game

In the 1970s, Robin Giles introduced a two-player dialogue game

You claim...	**I claim...**
$\varphi_1, \ldots, \varphi_n$	$\psi_1, \ldots, \psi_m$

consisting of two parts...

1. **Atomic statements** refer to experiments with a fixed probability of a positive result, and the players pay 1€ to their opponent for each incorrect statement – the winner *expects* not to lose money.
2. **Compound statements** are attacked or granted by the opposing player based on natural dialogue rules.

R. Giles. A non-classical logic for physics.
Studia Logica, 4(33):399–417 (1974).

R. Giles. Łukasiewicz logic and fuzzy set theory
International Journal of Man-Machine Studies, 8(3):313–327 (1976).

Elementary States

Atoms a, b are propositional variables p, q representing atomic statements, and the constant $\bot$ representing a statement that is always false.

Each atom a may be read as

> *"the (repeatable) elementary (yes/no) experiment E_a yields a positive result."*

Elementary states consist of a multiset of atoms $[a_1, \ldots, a_m]$ asserted by *you* and a multiset of atoms $[b_1, \ldots, b_n]$ asserted by *me*, written

$$[a_1, \ldots, a_m \parallel b_1, \ldots, b_n].$$

Elementary States

Atoms a, b are propositional variables p, q representing atomic statements, and the constant $\perp$ representing a statement that is always false.

Each atom a may be read as

> *“the (repeatable) elementary (yes/no) experiment E_a yields a positive result."*

Elementary states consist of a multiset of atoms $[a_1, \ldots, a_m]$ asserted by *you* and a multiset of atoms $[b_1, \ldots, b_n]$ asserted by *me*, written

$$[a_1, \ldots, a_m \,\|\, b_1, \ldots, b_n].$$

Elementary States

Atoms a, b are propositional variables p, q representing atomic statements, and the constant $\bot$ representing a statement that is always false.

Each atom a may be read as

> *"the (repeatable) elementary (yes/no) experiment E_a yields a positive result."*

Elementary states consist of a multiset of atoms $[a_1, \ldots, a_m]$ asserted by *you* and a multiset of atoms $[b_1, \ldots, b_n]$ asserted by *me*, written

$$[a_1, \ldots, a_m \parallel b_1, \ldots, b_n].$$

Risk

For every run of the game, a fixed **risk value** $\langle q \rangle \in [0, 1]$ is associated with each variable q, where $\langle \bot \rangle = 1$.

The **risk** associated with a multiset of atoms is then

$$\langle [a_1, \ldots, a_m] \rangle = \langle a_1 \rangle + \ldots + \langle a_m \rangle.$$

I.e., my risk corresponds to the amount that I *expect* to pay to you.

For an elementary state $[a_1, \ldots, a_m \mathbin{\|} b_1, \ldots, b_n]$,

$$\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$$

expresses that I do not expect any loss (possibly some gain).

Risk

For every run of the game, a fixed **risk value** $\langle q \rangle \in [0, 1]$ is associated with each variable q, where $\langle \bot \rangle = 1$.

The **risk** associated with a multiset of atoms is then

$$\langle [a_1, \ldots, a_m] \rangle = \langle a_1 \rangle + \ldots + \langle a_m \rangle.$$

I.e., my risk corresponds to the amount that I *expect* to pay to you.

For an elementary state $[a_1, \ldots, a_m \mathbin{\|} b_1, \ldots, b_n]$,

$$\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$$

expresses that I do not expect any loss (possibly some gain).

Risk

For every run of the game, a fixed **risk value** $\langle q \rangle \in [0, 1]$ is associated with each variable q, where $\langle \perp \rangle = 1$.

The **risk** associated with a multiset of atoms is then

$$\langle [a_1, \ldots, a_m] \rangle = \langle a_1 \rangle + \ldots + \langle a_m \rangle.$$

I.e., my risk corresponds to the amount that I *expect* to pay to you.

For an elementary state $[a_1, \ldots, a_m \mathbin{\|} b_1, \ldots, b_n]$,

$$\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$$

expresses that I do not expect any loss (possibly some gain).

An Example

Consider the elementary state

$$[p \parallel q, q].$$

The experiment E_p has to be performed once and E_q twice. If, e.g., all three outcomes are negative, then I owe you 2€ and you owe me 1€.

For $\langle p \rangle = \langle q \rangle = 0.5$, I expect an average *loss* of 0.5€.

For $\langle p \rangle = 0.8$ and $\langle q \rangle = 0.3$, I expect an average *gain* of 0.2€.

An Example

Consider the elementary state

$$[p \parallel q, q].$$

The experiment E_p has to be performed once and E_q twice. If, e.g., all three outcomes are negative, then I owe you 2€ and you owe me 1€.

For $\langle p \rangle = \langle q \rangle = 0.5$, I expect an average *loss* of 0.5€.

For $\langle p \rangle = 0.8$ and $\langle q \rangle = 0.3$, I expect an average *gain* of 0.2€.

An Example

Consider the elementary state

$$[p \parallel q, q].$$

The experiment E_p has to be performed once and E_q twice. If, e.g., all three outcomes are negative, then I owe you 2€ and you owe me 1€.

For $\langle p \rangle = \langle q \rangle = 0.5$, I expect an average *loss* of 0.5€.

For $\langle p \rangle = 0.8$ and $\langle q \rangle = 0.3$, I expect an average *gain* of 0.2€.

Compound Statements and Dialogue States

Compound statements are represented by **formulas** built (for now) from variables, the constant $\bot$, and the binary connective $\rightarrow$.

We can also consider the connectives $\wedge$, $\vee$, and $\odot$; however, in Łukasiewicz logic these are definable using $\rightarrow$ and $\bot$.

Dialogue states (d-states) consist of finite multisets $[\varphi_1, \ldots, \varphi_n]$ and $[\psi_1, \ldots, \psi_n]$ of formulas asserted by *you* and *me*, respectively, written

$$[\varphi_1, \ldots, \varphi_n \,\|\, \psi_1, \ldots, \psi_n].$$

Compound Statements and Dialogue States

Compound statements are represented by **formulas** built (for now) from variables, the constant $\bot$, and the binary connective $\to$.

We can also consider the connectives $\wedge$, $\vee$, and $\odot$; however, in Łukasiewicz logic these are definable using $\to$ and $\bot$.

Dialogue states (d-states) consist of finite multisets $[\varphi_1, \ldots, \varphi_n]$ and $[\psi_1, \ldots, \psi_n]$ of formulas asserted by *you* and *me*, respectively, written

$$[\varphi_1, \ldots, \varphi_n \parallel \psi_1, \ldots, \psi_n].$$

Compound Statements and Dialogue States

Compound statements are represented by **formulas** built (for now) from variables, the constant $\bot$, and the binary connective $\to$.

We can also consider the connectives $\wedge$, $\vee$, and $\odot$; however, in Łukasiewicz logic these are definable using $\to$ and $\bot$.

Dialogue states (d-states) consist of finite multisets $[\varphi_1, \ldots, \varphi_n]$ and $[\psi_1, \ldots, \psi_n]$ of formulas asserted by *you* and *me*, respectively, written

$$[\varphi_1, \ldots, \varphi_n \,\|\, \psi_1, \ldots, \psi_n].$$

Implication

The **dialogue rule for implication** is:

> *If I assert $\varphi \to \psi$, then whenever you choose to attack this statement by asserting φ, I must assert also ψ. (And vice versa, i.e., for the roles of me and you switched.)*
>
> *A player may also choose to* never *attack the opponent's assertion of $\varphi \to \psi$.*

Rounds

A **round** with **initiator** α and **respondent** β is a transition from one d-state to a **successor d-state** consisting of two **moves**:

1. α *chooses* one of the formulas $\varphi \to \psi$ asserted by β.
2. Either α *attacks* $\varphi \to \psi$ by asserting φ, and β must assert ψ, or α *grants* $\varphi \to \psi$ (will never attack that occurrence.)

   The occurrence of $\varphi \to \psi$ is removed from the assertions of β.

We make use of **intermediary states (i-states)**, denoting the initiator's choice of the formula that gets attacked or granted by underlining.

Rounds

A **round** with **initiator** α and **respondent** β is a transition from one d-state to a **successor d-state** consisting of two **moves**:

1. α *chooses* one of the formulas $\varphi \to \psi$ asserted by β.
2. Either α *attacks* $\varphi \to \psi$ by asserting φ, and β must assert ψ, or α *grants* $\varphi \to \psi$ (will never attack that occurrence.)

   The occurrence of $\varphi \to \psi$ is removed from the assertions of β.

We make use of **intermediary states (i-states)**, denoting the initiator's choice of the formula that gets attacked or granted by underlining.

Rounds

A **round** with **initiator** α and **respondent** β is a transition from one d-state to a **successor d-state** consisting of two **moves**:

1. α *chooses* one of the formulas $\varphi \to \psi$ asserted by β.
2. Either α *attacks* $\varphi \to \psi$ by asserting φ, and β must assert ψ, or α *grants* $\varphi \to \psi$ (will never attack that occurrence.)

   The occurrence of $\varphi \to \psi$ is removed from the assertions of β.

We make use of **intermediary states (i-states)**, denoting the initiator's choice of the formula that gets attacked or granted by underlining.

Rounds

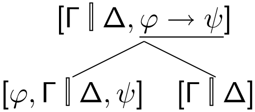
A **round** with **initiator** α and **respondent** β is a transition from one d-state to a **successor d-state** consisting of two **moves**:

1. α *chooses* one of the formulas $\varphi \to \psi$ asserted by β.
2. Either α *attacks* $\varphi \to \psi$ by asserting φ, and β must assert ψ, or α *grants* $\varphi \to \psi$ (will never attack that occurrence.)

   The occurrence of $\varphi \to \psi$ is removed from the assertions of β.

We make use of **intermediary states (i-states)**, denoting the initiator's choice of the formula that gets attacked or granted by underlining.

Implication Rules



$$[\Gamma \parallel \Delta, \underline{\varphi \to \psi}]$$

$$[\varphi, \Gamma \parallel \Delta, \psi] \quad [\Gamma \parallel \Delta]$$

$$[\underline{\varphi \to \psi}, \Gamma \parallel \Delta]$$

$$[\psi, \Gamma \parallel \Delta, \varphi]$$

$$[\underline{\varphi \to \psi}, \Gamma \parallel \Delta]$$

$$[\Gamma \parallel \Delta]$$

Whose Turn Is It?

A **regulation** ρ maps non-elementary d-states to a label **Y** or **I**, meaning “You / I initiate the next round."

A regulation is **consistent** if a d-state is mapped to **Y** (or **I**) only when an initiating move is possible for you (or me).

Whose Turn Is It?

A **regulation** ρ maps non-elementary d-states to a label **Y** or **I**, meaning “You / I initiate the next round."

A regulation is **consistent** if a d-state is mapped to **Y** (or **I**) only when an initiating move is possible for you (or me).

Game Forms and Games

A **game form** $\mathbf{G}([\Gamma \mid\mid \Delta], \rho)$ is a *tree of states* where

- the *root* is the initial d-state $[\Gamma \mid\mid \Delta]$
- the *successor nodes* to any state S are the states resulting from legal moves at S according to the consistent regulation ρ
- the *leaf nodes* are the reachable elementary states.

A **game** consists of a game form $\mathbf{G}([\Gamma \mid\mid \Delta], \rho)$ together with a risk assignment $\langle\cdot\rangle$, and a **run** of the game is a branch of $\mathbf{G}([\Gamma \mid\mid \Delta], \rho)$.

Game Forms and Games

A **game form** $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ is a *tree of states* where

- the *root* is the initial d-state $[\Gamma \parallel \Delta]$
- the *successor nodes* to any state S are the states resulting from legal moves at S according to the consistent regulation ρ
- the *leaf nodes* are the reachable elementary states.

A **game** consists of a game form $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ together with a risk assignment $\langle \cdot \rangle$, and a **run** of the game is a branch of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$.

Game Forms and Games

A **game form** $\mathbf{G}([\Gamma \mid\mid \Delta], \rho)$ is a *tree of states* where

- the *root* is the initial d-state $[\Gamma \mid\mid \Delta]$
- the *successor nodes* to any state *S* are the states resulting from legal moves at *S* according to the consistent regulation ρ
- the *leaf nodes* are the reachable elementary states.

A **game** consists of a game form $\mathbf{G}([\Gamma \mid\mid \Delta], \rho)$ together with a risk assignment $\langle\cdot\rangle$, and a **run** of the game is a branch of $\mathbf{G}([\Gamma \mid\mid \Delta], \rho)$.

Game Forms and Games

A **game form** $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ is a *tree of states* where

- the *root* is the initial d-state $[\Gamma \parallel \Delta]$
- the *successor nodes* to any state *S* are the states resulting from legal moves at *S* according to the consistent regulation ρ
- the *leaf nodes* are the reachable elementary states.

A **game** consists of a game form $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ together with a risk assignment $\langle \cdot \rangle$, and a **run** of the game is a branch of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$.

Game Forms and Games

A **game form** $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ is a *tree of states* where

- the *root* is the initial d-state $[\Gamma \parallel \Delta]$
- the *successor nodes* to any state *S* are the states resulting from legal moves at *S* according to the consistent regulation ρ
- the *leaf nodes* are the reachable elementary states.

A **game** consists of a game form $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ together with a risk assignment $\langle \cdot \rangle$, and a **run** of the game is a branch of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$.

Example

If it is *my* turn to move in the d-state $[p \to q \parallel a \to b, c \to d]$, then I must either attack or grant your statement $p \to q$, giving

$$\begin{array}{c} [p \to q \parallel a \to b, c \to d]^{\mathbf{I}} \\ | \\ [\underline{p \to q} \parallel a \to b, c \to d]^{\mathbf{I}} \\ | \\ [q \parallel p, a \to b, c \to d] \end{array} \quad \text{or} \quad \begin{array}{c} [p \to q \parallel a \to b, c \to d]^{\mathbf{I}} \\ | \\ [\underline{p \to q} \parallel a \to b, c \to d]^{\mathbf{I}} \\ | \\ [\ \parallel a \to b, c \to d]. \end{array}$$

Example

If it is *my* turn to move in the d-state $[p \to q \,\|\, a \to b, c \to d]$, then I must either attack or grant your statement $p \to q$, giving

$$\begin{tikzcd}
{[p \to q \,\|\, a \to b, c \to d]^{\mathrm{I}}} \arrow[d, dash] & \text{or} & {[p \to q \,\|\, a \to b, c \to d]^{\mathrm{I}}} \arrow[d, dash] \\
{[\underline{p \to q} \,\|\, a \to b, c \to d]^{\mathrm{I}}} \arrow[d, dash] & & {[\underline{p \to q} \,\|\, a \to b, c \to d]^{\mathrm{I}}} \arrow[d, dash] \\
{[q \,\|\, p, a \to b, c \to d]} & & {[\;\|\, a \to b, c \to d].}
\end{tikzcd}$$

Example (Continutcd)

If it is *your* turn to move, there are four possibilities:

$$\begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, \underline{a \to b}, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q, a \,\|\, b, c \to d] \end{array} \quad \text{or} \quad \begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, \underline{a \to b}, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, c \to d] \end{array}$$

$$\text{or} \quad \begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, a \to b, \underline{c \to d}]^{\mathbf{Y}} \\ | \\ [p \to q, c \,\|\, a \to b, d] \end{array} \quad \text{or} \quad \begin{array}{c} [p \to q \,\|\, a \to b, c \to d]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, a \to b, \underline{c \to d}]^{\mathbf{Y}} \\ | \\ [p \to q \,\|\, a \to b]. \end{array}$$

Winning

Suppose that a run of $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ with risk assignment $\langle \cdot \rangle$ ends with the elementary state $[a_1, \ldots, a_m \parallel b_1, \ldots, b_n]$.

I **win** in that run if I do not expect any loss resulting from betting on the corresponding elementary experiments, i.e., if

$$\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle.$$

Winning

Suppose that a run of $\mathbf{G}([\Gamma \mathrel{\|} \Delta], \rho)$ with risk assignment $\langle\cdot\rangle$ ends with the elementary state $[a_1, \ldots, a_m \mathrel{\|} b_1, \ldots, b_n]$.

I **win** in that run if I do not expect any loss resulting from betting on the corresponding elementary experiments, i.e., if

$$\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle.$$

Strategies

A **strategy (for me)** is obtained from a game form by (iteratively from the root) deleting all but one successor of every state labelled **I**.

A strategy for a game form $\mathbf{G}([\Gamma \mathrel{\|} \Delta], \rho)$ is a **winning strategy (for me) for a risk assignment** $\langle\cdot\rangle$ if $\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$ holds for each of its leaf nodes $[a_1, \ldots, a_m \mathrel{\|} b_1, \ldots, b_n]$.

Strategies

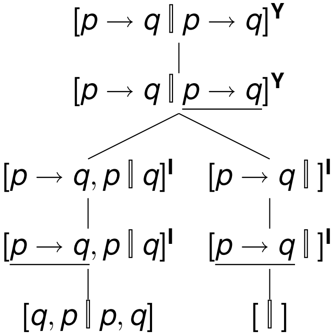
A **strategy (for me)** is obtained from a game form by (iteratively from the root) deleting all but one successor of every state labelled **I**.

A strategy for a game form $\mathbf{G}([\Gamma \mathrel{\|} \Delta], \rho)$ is a **winning strategy (for me) for a risk assignment** $\langle\cdot\rangle$ if $\langle a_1, \ldots, a_m\rangle \geq \langle b_1, \ldots, b_n\rangle$ holds for each of its leaf nodes $[a_1, \ldots, a_m \mathrel{\|} b_1, \ldots, b_n]$.

Example (1)

Consider a game form $\mathbf{G}([p \to q \parallel p \to q], \rho)$.

If $\rho([p \to q \parallel p \to q]) = \mathbf{Y}$, then the strategy



$$[p \to q \parallel p \to q]^{\mathbf{Y}}$$

$$[p \to q \parallel \underline{p \to q}]^{\mathbf{Y}}$$

$$[p \to q, p \parallel q]^{\mathbf{I}} \qquad [p \to q \parallel \,]^{\mathbf{I}}$$

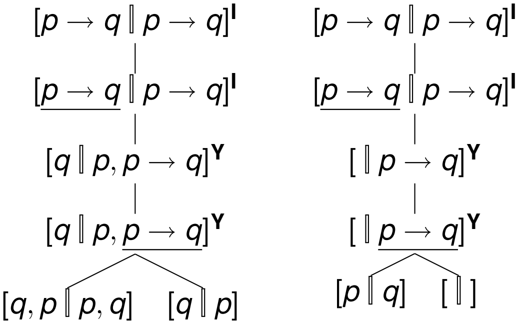
$$[\underline{p \to q}, p \parallel q]^{\mathbf{I}} \qquad [\underline{p \to q} \parallel \,]^{\mathbf{I}}$$

$$[q, p \parallel p, q] \qquad [\,\parallel\,]$$

is winning for *any* risk assignment $\langle \cdot \rangle$

Example (2)

However, if $\rho([p \to q \parallel p \to q]) = \mathbf{I}$, then the strategies



$$[p \to q \parallel p \to q]^{\mathbf{I}}$$

$$[\underline{p \to q} \parallel p \to q]^{\mathbf{I}}$$

$$[q \parallel p, p \to q]^{\mathbf{Y}}$$

$$[q \parallel p, \underline{p \to q}]^{\mathbf{Y}}$$

$$[q, p \parallel p, q] \qquad [q \parallel p]$$

$$[p \to q \parallel p \to q]^{\mathbf{I}}$$

$$[\underline{p \to q} \parallel p \to q]^{\mathbf{I}}$$

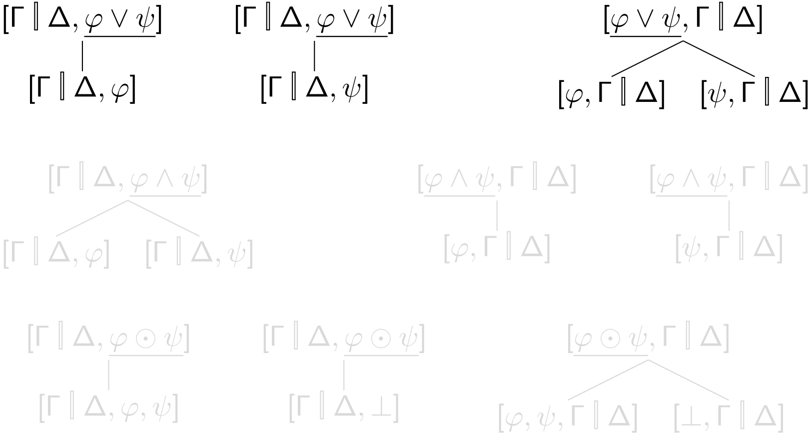
$$[\ \parallel p \to q]^{\mathbf{Y}}$$

$$[\ \parallel \underline{p \to q}]^{\mathbf{Y}}$$

$$[p \parallel q] \qquad [\ \parallel\ ]$$

are winning only if $\langle q \rangle \geq \langle p \rangle$ and $\langle p \rangle \geq \langle q \rangle$, respectively.

Other Connectives



$[\Gamma \parallel \Delta, \underline{\varphi \vee \psi}]$ — $[\Gamma \parallel \Delta, \varphi]$

$[\Gamma \parallel \Delta, \underline{\varphi \vee \psi}]$ — $[\Gamma \parallel \Delta, \psi]$

$[\underline{\varphi \vee \psi}, \Gamma \parallel \Delta]$ — $[\varphi, \Gamma \parallel \Delta]$ $[\psi, \Gamma \parallel \Delta]$

$[\Gamma \parallel \Delta, \underline{\varphi \wedge \psi}]$ — $[\Gamma \parallel \Delta, \varphi]$ $[\Gamma \parallel \Delta, \psi]$

$[\underline{\varphi \wedge \psi}, \Gamma \parallel \Delta]$ — $[\varphi, \Gamma \parallel \Delta]$

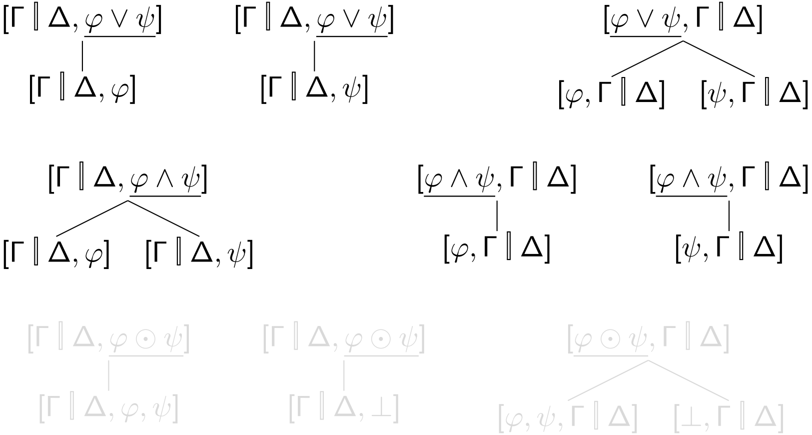
$[\underline{\varphi \wedge \psi}, \Gamma \parallel \Delta]$ — $[\psi, \Gamma \parallel \Delta]$

$[\Gamma \parallel \Delta, \underline{\varphi \odot \psi}]$ — $[\Gamma \parallel \Delta, \varphi, \psi]$

$[\Gamma \parallel \Delta, \underline{\varphi \odot \psi}]$ — $[\Gamma \parallel \Delta, \bot]$

$[\underline{\varphi \odot \psi}, \Gamma \parallel \Delta]$ — $[\varphi, \psi, \Gamma \parallel \Delta]$ $[\bot, \Gamma \parallel \Delta]$

Other Connectives



$[\Gamma \parallel \Delta, \underline{\varphi \vee \psi}]$

$[\Gamma \parallel \Delta, \varphi]$

$[\Gamma \parallel \Delta, \underline{\varphi \vee \psi}]$

$[\Gamma \parallel \Delta, \psi]$

$[\underline{\varphi \vee \psi}, \Gamma \parallel \Delta]$

$[\varphi, \Gamma \parallel \Delta]$ $[\psi, \Gamma \parallel \Delta]$

$[\Gamma \parallel \Delta, \underline{\varphi \wedge \psi}]$

$[\Gamma \parallel \Delta, \varphi]$ $[\Gamma \parallel \Delta, \psi]$

$[\underline{\varphi \wedge \psi}, \Gamma \parallel \Delta]$

$[\varphi, \Gamma \parallel \Delta]$

$[\underline{\varphi \wedge \psi}, \Gamma \parallel \Delta]$

$[\psi, \Gamma \parallel \Delta]$

$[\Gamma \parallel \Delta, \underline{\varphi \odot \psi}]$

$[\Gamma \parallel \Delta, \varphi, \psi]$

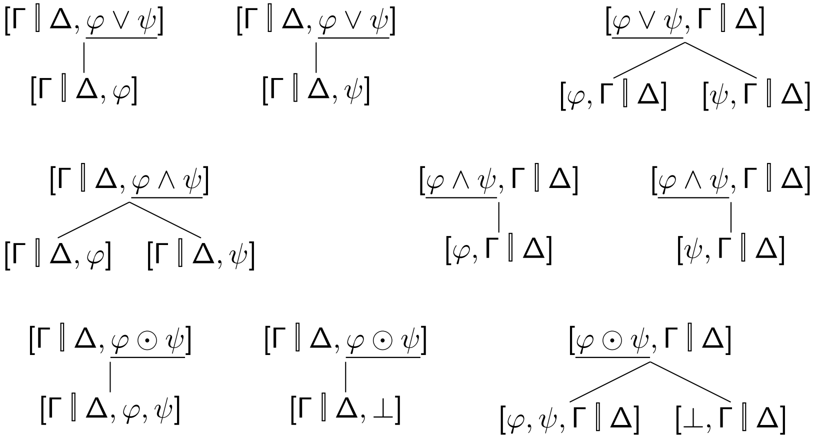
$[\Gamma \parallel \Delta, \underline{\varphi \odot \psi}]$

$[\Gamma \parallel \Delta, \bot]$

$[\underline{\varphi \odot \psi}, \Gamma \parallel \Delta]$

$[\varphi, \psi, \Gamma \parallel \Delta]$ $[\bot, \Gamma \parallel \Delta]$

Other Connectives



$$[\Gamma \parallel \Delta, \underline{\varphi \vee \psi}]$$
$$[\Gamma \parallel \Delta, \varphi]$$

$$[\Gamma \parallel \Delta, \underline{\varphi \vee \psi}]$$
$$[\Gamma \parallel \Delta, \psi]$$

$$[\underline{\varphi \vee \psi}, \Gamma \parallel \Delta]$$
$$[\varphi, \Gamma \parallel \Delta] \quad [\psi, \Gamma \parallel \Delta]$$

$$[\Gamma \parallel \Delta, \underline{\varphi \wedge \psi}]$$
$$[\Gamma \parallel \Delta, \varphi] \quad [\Gamma \parallel \Delta, \psi]$$

$$[\underline{\varphi \wedge \psi}, \Gamma \parallel \Delta]$$
$$[\varphi, \Gamma \parallel \Delta]$$

$$[\underline{\varphi \wedge \psi}, \Gamma \parallel \Delta]$$
$$[\psi, \Gamma \parallel \Delta]$$

$$[\Gamma \parallel \Delta, \underline{\varphi \odot \psi}]$$
$$[\Gamma \parallel \Delta, \varphi, \psi]$$

$$[\Gamma \parallel \Delta, \underline{\varphi \odot \psi}]$$
$$[\Gamma \parallel \Delta, \bot]$$

$$[\underline{\varphi \odot \psi}, \Gamma \parallel \Delta]$$
$$[\varphi, \psi, \Gamma \parallel \Delta] \quad [\bot, \Gamma \parallel \Delta]$$

Łukasiewicz Logic

- **Łukasiewicz logic** Ł is an infinite-valued logic introduced by Jan Łukasiewicz in the 1920s, now considered to be one of the "fundamental fuzzy logics".

  J. Łukasiewicz and A. Tarski. Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Societé des Sciences et des Lettres de Varsovie, Classe III*, 23, 1930.

- Ł and its algebraic semantics **MV-algebras** enjoy close relationships with lattice-ordered abelian groups, rational polyhedra, C*-algebras, Ulam and Giles games, etc.

Łukasiewicz Logic

- **Łukasiewicz logic** Ł is an infinite-valued logic introduced by Jan Łukasiewicz in the 1920s, now considered to be one of the "fundamental fuzzy logics".

  J. Łukasiewicz and A. Tarski. Untersuchungen über den Aussagenkalkül. *Comptes Rendus des Séances de la Societé des Sciences et des Lettres de Varsovie, Classe III*, 23, 1930.

- Ł and its algebraic semantics **MV-algebras** enjoy close relationships with lattice-ordered abelian groups, rational polyhedra, C*-algebras, Ulam and Giles games, etc.

Syntax and Semantics

Formulas are built using → and ⊥, and we also define:

$$\begin{array}{rclcrcl} \neg\varphi & = & \varphi \to \bot & \qquad & \varphi \odot \psi & = & \neg(\varphi \to \neg\psi) \\ \varphi \vee \psi & = & (\varphi \to \psi) \to \psi & \qquad & \varphi \wedge \psi & = & \neg(\neg\varphi \vee \neg\psi). \end{array}$$

An **Ł-valuation** is a function v from formulas to $[0,1]$ satisfying

$$v(\bot) = 0 \quad \text{and} \quad v(\varphi \to \psi) = \min(1, 1 - v(\varphi) + v(\psi))$$

where also, by calculation

$$\begin{array}{rclcrcl} v(\neg\varphi) & = & 1 - v(\varphi) & \qquad & \varphi \odot \psi & = & \max(0, v(\varphi) + v(\psi) - 1) \\ v(\varphi \vee \psi) & = & \max(v(\varphi), v(\psi)) & \qquad & \varphi \wedge \psi & = & \min(v(\varphi), v(\psi)). \end{array}$$

A formula φ is **Ł-valid** if $v(\varphi) = 1$ for all Ł-valuations v.

Syntax and Semantics

Formulas are built using → and ⊥, and we also define:

$$\begin{array}{rcl} \neg\varphi & = & \varphi \to \bot \\ \varphi \vee \psi & = & (\varphi \to \psi) \to \psi \end{array} \qquad \begin{array}{rcl} \varphi \odot \psi & = & \neg(\varphi \to \neg\psi) \\ \varphi \wedge \psi & = & \neg(\neg\varphi \vee \neg\psi). \end{array}$$

An **Ł-valuation** is a function v from formulas to $[0, 1]$ satisfying

$$v(\bot) = 0 \quad \text{and} \quad v(\varphi \to \psi) = \min(1, 1 - v(\varphi) + v(\psi))$$

where also, by calculation

$$\begin{array}{rcl} v(\neg\varphi) & = & 1 - v(\varphi) \\ v(\varphi \vee \psi) & = & \max(v(\varphi), v(\psi)) \end{array} \qquad \begin{array}{rcl} \varphi \odot \psi & = & \max(0, v(\varphi) + v(\psi) - 1) \\ \varphi \wedge \psi & = & \min(v(\varphi), v(\psi)). \end{array}$$

A formula φ is **Ł-valid** if $v(\varphi) = 1$ for all Ł-valuations v.

Syntax and Semantics

Formulas are built using → and $\bot$, and we also define:

$$\begin{array}{rcl} \neg\varphi & = & \varphi \to \bot \\ \varphi \vee \psi & = & (\varphi \to \psi) \to \psi \end{array} \qquad \begin{array}{rcl} \varphi \odot \psi & = & \neg(\varphi \to \neg\psi) \\ \varphi \wedge \psi & = & \neg(\neg\varphi \vee \neg\psi). \end{array}$$

An **Ł-valuation** is a function v from formulas to $[0,1]$ satisfying

$$v(\bot) = 0 \quad \text{and} \quad v(\varphi \to \psi) = \min(1, 1 - v(\varphi) + v(\psi))$$

where also, by calculation

$$\begin{array}{rcl} v(\neg\varphi) & = & 1 - v(\varphi) \\ v(\varphi \vee \psi) & = & \max(v(\varphi), v(\psi)) \end{array} \qquad \begin{array}{rcl} \varphi \odot \psi & = & \max(0, v(\varphi) + v(\psi) - 1) \\ \varphi \wedge \psi & = & \min(v(\varphi), v(\psi)). \end{array}$$

A formula φ is **Ł-valid** if $v(\varphi) = 1$ for all Ł-valuations v.

Giles and Łukasiewicz

Theorem (Giles)

The following are equivalent for any formula φ:

1. φ *is Ł-valid.*
2. *I have a winning strategy for the game* $\mathbf{G}([\;\|\;\varphi], \rho)$ *with any risk assignment* $\langle\cdot\rangle$*, where* ρ *is an arbitrary consistent regulation.*

Disjunctive Strategies

A **state disjunction** is written

$$D = S_1 \bigvee \ldots \bigvee S_n.$$

A **disjunctive strategy** for D respecting a regulation ρ is a tree of state disjunctions with root D and two kinds of non-leaf nodes

1. **Playing nodes**, focussed on some component S_i of D, where the successor nodes are like those for S_i in strategies, except for the presence of additional components (that remain unchanged).
2. **Duplicating nodes**, where the single successor node is obtained by duplicating one of the components in D.

Disjunctive Strategies

A **state disjunction** is written

$$D = S_1 \bigvee \ldots \bigvee S_n.$$

A **disjunctive strategy** for D respecting a regulation ρ is a tree of state disjunctions with root D and two kinds of non-leaf nodes

1. **Playing nodes**, focussed on some component S_i of D, where the successor nodes are like those for S_i in strategies, except for the presence of additional components (that remain unchanged).
2. **Duplicating nodes**, where the single successor node is obtained by duplicating one of the components in D.

Disjunctive Strategies

A **state disjunction** is written

$$D = S_1 \bigvee \ldots \bigvee S_n.$$

A **disjunctive strategy** for D respecting a regulation ρ is a tree of state disjunctions with root D and two kinds of non-leaf nodes

1. **Playing nodes**, focussed on some component S_i of D, where the successor nodes are like those for S_i in strategies, except for the presence of additional components (that remain unchanged).
2. **Duplicating nodes**, where the single successor node is obtained by duplicating one of the components in D.

Disjunctive Strategies

A **state disjunction** is written

$$D = S_1 \bigvee \ldots \bigvee S_n.$$

A **disjunctive strategy** for D respecting a regulation ρ is a tree of state disjunctions with root D and two kinds of non-leaf nodes

1. **Playing nodes**, focussed on some component S_i of D, where the successor nodes are like those for S_i in strategies, except for the presence of additional components (that remain unchanged).
2. **Duplicating nodes**, where the single successor node is obtained by duplicating one of the components in D.

Disjunctive Winning Strategies

A disjunction of elementary states D is **winning (for me)** if for *every* risk assignment $\langle\cdot\rangle$

$$\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$$

for some $[a_1, \ldots, a_m \mathrel{\|} b_1, \ldots, b_n]$ in D.

A **disjunctive winning strategy (for me)** for $\mathbf{G}([\Gamma \mathrel{\|} \Delta], \rho)$ is a disjunctive strategy such that every leaf node is winning.

Disjunctive Winning Strategies

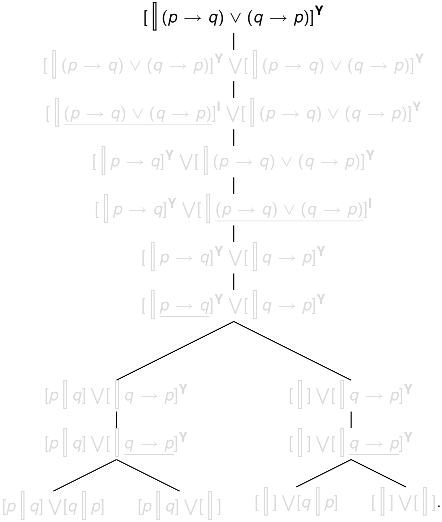
A disjunction of elementary states D is **winning (for me)** if for *every* risk assignment $\langle\cdot\rangle$

$$\langle a_1, \ldots, a_m \rangle \geq \langle b_1, \ldots, b_n \rangle$$

for some $[a_1, \ldots, a_m \,\|\, b_1, \ldots, b_n]$ in D.

A **disjunctive winning strategy (for me)** for $\mathbf{G}([\Gamma \,\|\, \Delta], \rho)$ is a disjunctive strategy such that every leaf node is winning.

Example



$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$

$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$

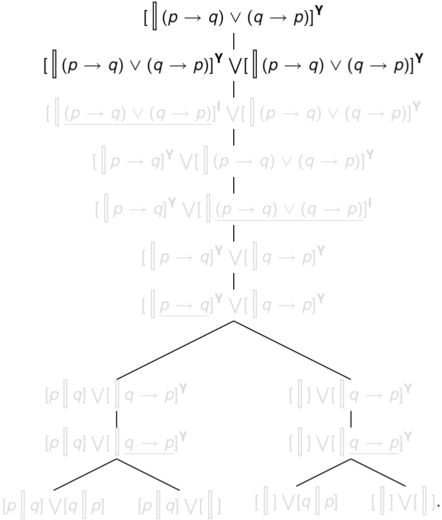
$[\,\| \underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$

$[p \| q] \bigvee [\,\| q \to p]^{\mathbf{Y}}$ $\qquad$ $[\,\|\,] \bigvee [\,\| q \to p]^{\mathbf{Y}}$

$[p \| q] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}}$ $\qquad$ $[\,\|\,] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}}$

$[p \| q] \bigvee [q \| p]$ $\qquad$ $[p \| q] \bigvee [\,\|\,]$ $\qquad$ $[\,\|\,] \bigvee [q \| p]$ $\qquad$ $[\,\|\,] \bigvee [\,\|\,]$.

Example



$$[\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,\underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,\underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

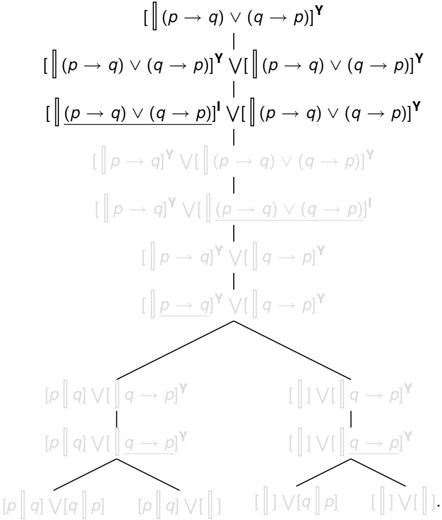
$$[\,\|\,\underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [\,\|\,q \to p]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [\,\|\,\underline{q \to p}]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\|\,\underline{q \to p}]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [q\,\|\,p] \qquad [p\,\|\,q] \bigvee [\,\|\,] \qquad [\,\|\,] \bigvee [q\,\|\,p] \qquad [\,\|\,] \bigvee [\,\|\,].$$

Example



$[\,\Vert\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\,(p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\Vert\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\,\underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\Vert\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\,p \to q]^{\mathbf{Y}} \bigvee [\,\Vert\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\,p \to q]^{\mathbf{Y}} \bigvee [\,\Vert\,\underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$

$[\,\Vert\,p \to q]^{\mathbf{Y}} \bigvee [\,\Vert\,q \to p]^{\mathbf{Y}}$

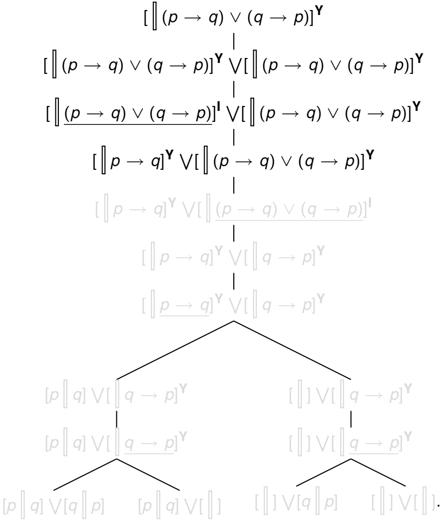
$[\,\Vert\,\underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\Vert\,q \to p]^{\mathbf{Y}}$

$[p\,\Vert\,q] \bigvee [\,\Vert\,q \to p]^{\mathbf{Y}}$ $\qquad$ $[\,\Vert\,] \bigvee [\,\Vert\,q \to p]^{\mathbf{Y}}$

$[p\,\Vert\,q] \bigvee [\,\Vert\,\underline{q \to p}]^{\mathbf{Y}}$ $\qquad$ $[\,\Vert\,] \bigvee [\,\Vert\,\underline{q \to p}]^{\mathbf{Y}}$

$[p\,\Vert\,q] \bigvee [q\,\Vert\,p]$ $\qquad$ $[p\,\Vert\,q] \bigvee [\,\Vert\,]$ $\qquad$ $[\,\Vert\,] \bigvee [q\,\Vert\,p]$ $\qquad$ $[\,\Vert\,] \bigvee [\,\Vert\,].$

Example



$[\ \Vert (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\ \Vert (p \to q) \lor (q \to p)]^{\mathbf{Y}} \bigvee [\ \Vert (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\ \Vert \underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}} \bigvee [\ \Vert (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\ \Vert p \to q]^{\mathbf{Y}} \bigvee [\ \Vert (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\ \Vert p \to q]^{\mathbf{Y}} \bigvee [\ \Vert \underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}}$

$[\ \Vert p \to q]^{\mathbf{Y}} \bigvee [\ \Vert q \to p]^{\mathbf{Y}}$

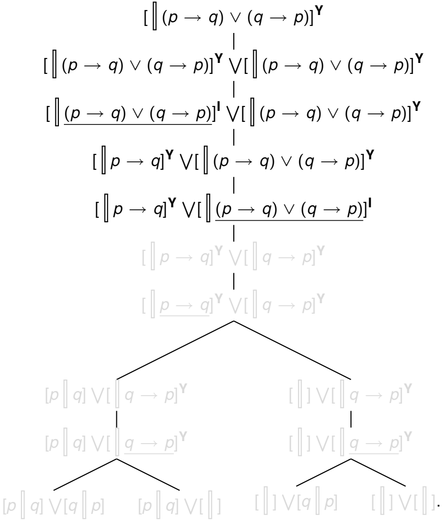
$[\ \Vert \underline{p \to q}]^{\mathbf{Y}} \bigvee [\ \Vert q \to p]^{\mathbf{Y}}$

$[p \Vert q] \bigvee [\ \Vert q \to p]^{\mathbf{Y}}$ $\qquad$ $[\ \Vert\ ] \bigvee [\ \Vert q \to p]^{\mathbf{Y}}$

$[p \Vert q] \bigvee [\ \Vert \underline{q \to p}]^{\mathbf{Y}}$ $\qquad$ $[\ \Vert\ ] \bigvee [\ \Vert \underline{q \to p}]^{\mathbf{Y}}$

$[p \Vert q] \bigvee [q \Vert p]$ $\qquad$ $[p \Vert q] \bigvee [\ \Vert\ ]$ $\qquad$ $[\ \Vert\ ] \bigvee [q \Vert p]$ $\qquad$ $[\ \Vert\ ] \bigvee [\ \Vert\ ].$

Example



$$[\,\|\,(p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,(p \to q) \lor (q \to p)]^{\mathbf{Y}} \bigvee [\,\|\,(p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,\underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}} \bigvee [\,\|\,(p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,(p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,\underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

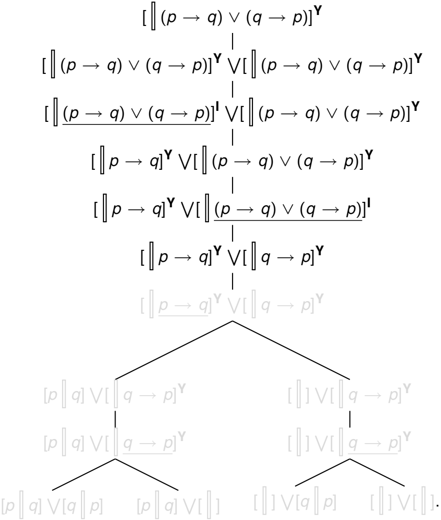
$$[\,\|\,\underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [\,\|\,q \to p]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [\,\|\,\underline{q \to p}]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\|\,\underline{q \to p}]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [q\,\|\,p] \qquad [p\,\|\,q] \bigvee [\,\|\,] \qquad [\,\|\,] \bigvee [q\,\|\,p] \qquad [\,\|\,] \bigvee [\,\|\,].$$

Example



$[\,\Vert\, (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\, (p \to q) \lor (q \to p)]^{\mathbf{Y}} \bigvee [\,\Vert\, (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\, \underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}} \bigvee [\,\Vert\, (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\, p \to q]^{\mathbf{Y}} \bigvee [\,\Vert\, (p \to q) \lor (q \to p)]^{\mathbf{Y}}$

$[\,\Vert\, p \to q]^{\mathbf{Y}} \bigvee [\,\Vert\, \underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}}$

$[\,\Vert\, p \to q]^{\mathbf{Y}} \bigvee [\,\Vert\, q \to p]^{\mathbf{Y}}$

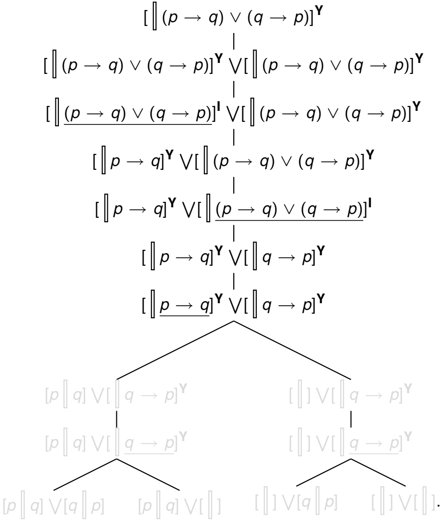
$[\,\Vert\, \underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\Vert\, q \to p]^{\mathbf{Y}}$

$[p \Vert q] \bigvee [\,\Vert\, q \to p]^{\mathbf{Y}}$ $\quad$ $[\,\Vert\,] \bigvee [\,\Vert\, q \to p]^{\mathbf{Y}}$

$[p \Vert q] \bigvee [\,\Vert\, \underline{q \to p}]^{\mathbf{Y}}$ $\quad$ $[\,\Vert\,] \bigvee [\,\Vert\, \underline{q \to p}]^{\mathbf{Y}}$

$[p \Vert q] \bigvee [q \Vert p]$ $\quad$ $[p \Vert q] \bigvee [\,\Vert\,]$ $\quad$ $[\,\Vert\,] \bigvee [q \Vert p]$ $\quad$ $[\,\Vert\,] \bigvee [\,\Vert\,].$

Example



$$[\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel p \to q]^{\mathbf{Y}} \bigvee [\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel p \to q]^{\mathbf{Y}} \bigvee [\parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\parallel p \to q]^{\mathbf{Y}} \bigvee [\parallel q \to p]^{\mathbf{Y}}$$

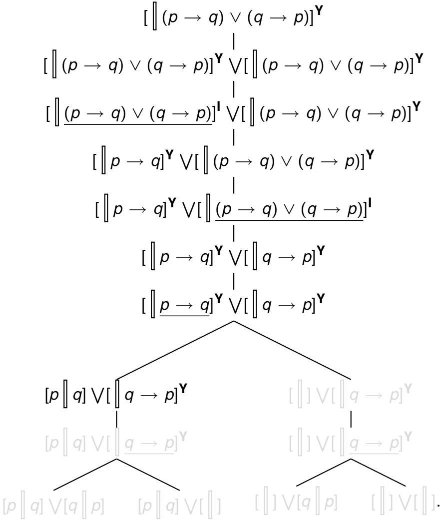
$$[\parallel \underline{p \to q}]^{\mathbf{Y}} \bigvee [\parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\parallel q \to p]^{\mathbf{Y}} \qquad [\parallel] \bigvee [\parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\parallel \underline{q \to p}]^{\mathbf{Y}} \qquad [\parallel] \bigvee [\parallel \underline{q \to p}]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [q \parallel p] \qquad [p \parallel q] \bigvee [\parallel] \qquad [\parallel] \bigvee [q \parallel p] \qquad [\parallel] \bigvee [\parallel].$$

Example



$$[\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel p \to q]^{\mathbf{Y}} \bigvee [\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\parallel p \to q]^{\mathbf{Y}} \bigvee [\parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\parallel p \to q]^{\mathbf{Y}} \bigvee [\parallel q \to p]^{\mathbf{Y}}$$

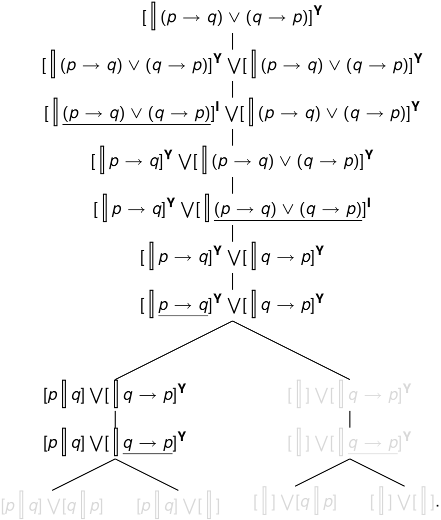
$$[\parallel \underline{p \to q}]^{\mathbf{Y}} \bigvee [\parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\parallel q \to p]^{\mathbf{Y}} \qquad [\parallel] \bigvee [\parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\parallel \underline{q \to p}]^{\mathbf{Y}} \qquad [\parallel] \bigvee [\parallel \underline{q \to p}]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [q \parallel p] \qquad [p \parallel q] \bigvee [\parallel] \qquad [\parallel] \bigvee [q \parallel p] \qquad [\parallel] \bigvee [\parallel].$$

Example



$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$

$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$

$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$

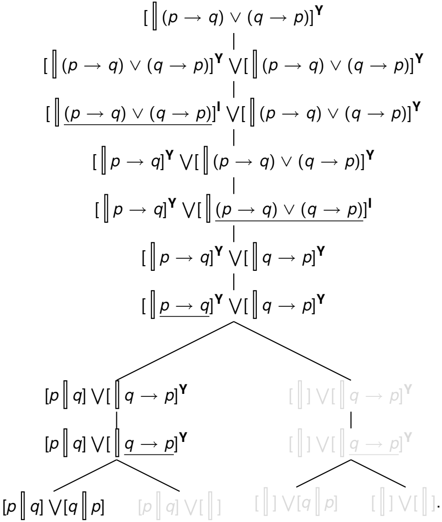
$[\,\| \underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$

$[p \| q] \bigvee [\,\| q \to p]^{\mathbf{Y}}$ $\qquad$ $[\,\|\,] \bigvee [\,\| q \to p]^{\mathbf{Y}}$

$[p \| q] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}}$ $\qquad$ $[\,\|\,] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}}$

$[p \| q] \bigvee [q \| p]$ $\qquad$ $[p \| q] \bigvee [\,\|\,]$ $\qquad$ $[\,\|\,] \bigvee [q \| p]$ $\qquad$ $[\,\|\,] \bigvee [\,\|\,].$

Example



$$[\,\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\parallel p \to q]^{\mathbf{Y}} \bigvee [\,\parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\parallel p \to q]^{\mathbf{Y}} \bigvee [\,\parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\,\parallel p \to q]^{\mathbf{Y}} \bigvee [\,\parallel q \to p]^{\mathbf{Y}}$$

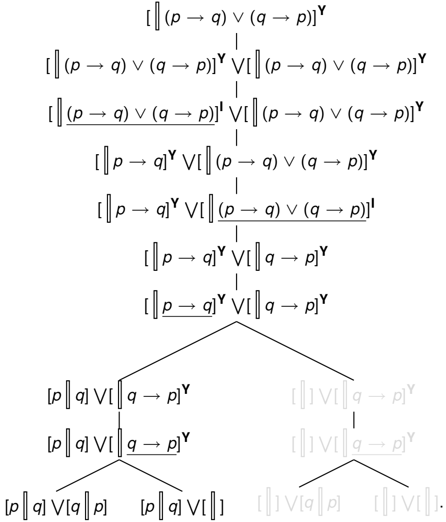
$$[\,\parallel \underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\,\parallel q \to p]^{\mathbf{Y}} \qquad [\,\parallel\,] \bigvee [\,\parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\,\parallel \underline{q \to p}]^{\mathbf{Y}} \qquad [\,\parallel\,] \bigvee [\,\parallel \underline{q \to p}]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [q \parallel p] \qquad [p \parallel q] \bigvee [\,\parallel\,] \qquad [\,\parallel\,] \bigvee [q \parallel p] \qquad [\,\parallel\,] \bigvee [\,\parallel\,].$$

Example



$$[\,\| (p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\| (p \to q) \lor (q \to p)]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\| \underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}} \bigvee [\,\| (p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \lor (q \to p)]^{\mathbf{Y}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| \underline{(p \to q) \lor (q \to p)}]^{\mathbf{I}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

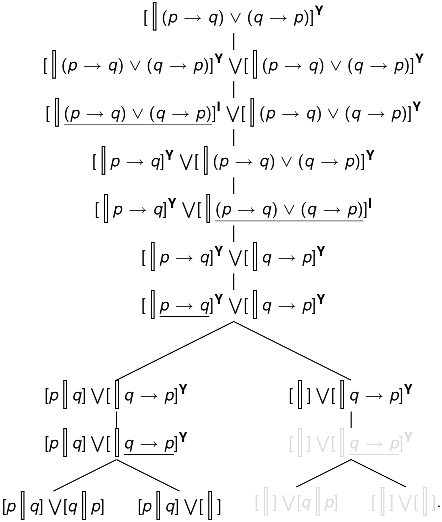
$$[\,\| \underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [\,\| q \to p]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [q \| p] \qquad [p \| q] \bigvee [\,\|\,] \qquad [\,\|\,] \bigvee [q \| p] \qquad [\,\|\,] \bigvee [\,\|\,].$$

Example



$$[\ \parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\ \parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\ \parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\ \parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\ \parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\ \parallel p \to q]^{\mathbf{Y}} \bigvee [\ \parallel (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\ \parallel p \to q]^{\mathbf{Y}} \bigvee [\ \parallel \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\ \parallel p \to q]^{\mathbf{Y}} \bigvee [\ \parallel q \to p]^{\mathbf{Y}}$$

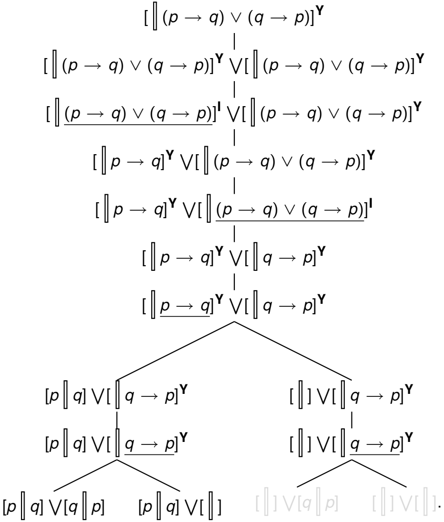
$$[\ \parallel \underline{p \to q}]^{\mathbf{Y}} \bigvee [\ \parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\ \parallel q \to p]^{\mathbf{Y}} \qquad [\ \parallel\ ] \bigvee [\ \parallel q \to p]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [\ \parallel \underline{q \to p}]^{\mathbf{Y}} \qquad [\ \parallel\ ] \bigvee [\ \parallel \underline{q \to p}]^{\mathbf{Y}}$$

$$[p \parallel q] \bigvee [q \parallel p] \qquad [p \parallel q] \bigvee [\ \parallel\ ] \qquad [\ \parallel\ ] \bigvee [q \parallel p] \qquad [\ \parallel\ ] \bigvee [\ \parallel\ ].$$

Example



$$[\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,\underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,(p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,\underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\,\|\,p \to q]^{\mathbf{Y}} \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

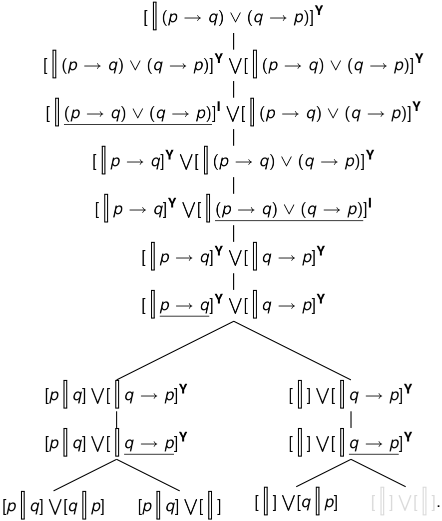
$$[\,\|\,\underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [\,\|\,q \to p]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\|\,q \to p]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [\,\|\,\underline{q \to p}]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\|\,\underline{q \to p}]^{\mathbf{Y}}$$

$$[p\,\|\,q] \bigvee [q\,\|\,p] \qquad [p\,\|\,q] \bigvee [\,\|\,] \qquad [\,\|\,] \bigvee [q\,\|\,p] \qquad [\,\|\,] \bigvee [\,\|\,].$$

Example



$$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

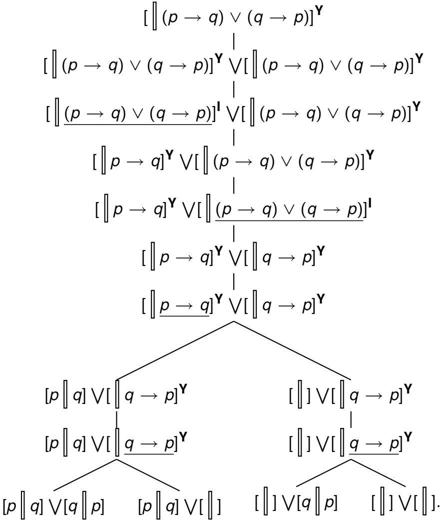
$$[\,\| \underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [\,\| q \to p]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [q \| p] \qquad [p \| q] \bigvee [\,\|\,] \qquad [\,\|\,] \bigvee [q \| p] \qquad [\,\|\,] \bigvee [\,\|\,].$$

Example



$$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| (p \to q) \vee (q \to p)]^{\mathbf{Y}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| \underline{(p \to q) \vee (q \to p)}]^{\mathbf{I}}$$

$$[\,\| p \to q]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

$$[\,\| \underline{p \to q}]^{\mathbf{Y}} \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [\,\| q \to p]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\| q \to p]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}} \qquad [\,\|\,] \bigvee [\,\| \underline{q \to p}]^{\mathbf{Y}}$$

$$[p \| q] \bigvee [q \| p] \qquad [p \| q] \bigvee [\,\|\,] \qquad [\,\|\,] \bigvee [q \| p] \qquad [\,\|\,] \bigvee [\,\|\,].$$

Giles and Łukasiewicz Again

Theorem (Fermüller and Metcalfe)

The following are equivalent for any formula φ:

1. φ *is Ł-valid.*
2. *I have a disjunctive winning strategy for the game* $\mathbf{G}([\ \| \varphi], \rho)$ *for an arbitrary consistent regulation* ρ.

Sequents

A **sequent** is an ordered pair of finite multisets of formulas Γ and Δ, written $\Gamma \Rightarrow \Delta$ (essentially, a *dialogue state*).

The following **sequent rules** represent elements of a strategy:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

Let SŁ be the **sequent calculus** consisting of these rules plus

$$\frac{}{\Gamma, \underbrace{\bot, \ldots, \bot}_{n}, \Delta \Rightarrow \Delta, \varphi_1, \ldots, \varphi_n} \qquad \frac{\Gamma, \varphi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

Theorem (Adamson and Giles)

φ is Ł-valid iff $\Rightarrow \varphi$ is derivable in SŁ.

A. Adamson and R. Giles. A Game-Based Formal System for $Ł_\infty$. *Studia Logica* 1(38) (1979), 49–73.

Sequents

A **sequent** is an ordered pair of finite multisets of formulas Γ and Δ, written $\Gamma \Rightarrow \Delta$ (essentially, a *dialogue state*).

The following **sequent rules** represent elements of a strategy:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

Let SŁ be the **sequent calculus** consisting of these rules plus

$$\frac{}{\Gamma, \underbrace{\bot, \ldots, \bot}_{n}, \Delta \Rightarrow \Delta, \varphi_1, \ldots, \varphi_n} \qquad \frac{\Gamma, \varphi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

Theorem (Adamson and Giles)

φ *is Ł-valid* *iff* $\Rightarrow \varphi$ *is derivable in* SŁ.

A. Adamson and R. Giles. A Game-Based Formal System for Ł$_\infty$. *Studia Logica* 1(38) (1979), 49–73.

Sequents

A **sequent** is an ordered pair of finite multisets of formulas Γ and Δ, written $\Gamma \Rightarrow \Delta$ (essentially, a *dialogue state*).

The following **sequent rules** represent elements of a strategy:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

Let SŁ be the **sequent calculus** consisting of these rules plus

$$\frac{}{\Gamma, \underbrace{\bot, \ldots, \bot}_{n}, \Delta \Rightarrow \Delta, \varphi_1, \ldots, \varphi_n} \qquad \frac{\Gamma, \varphi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

Theorem (Adamson and Giles)

φ is Ł-valid iff $\Rightarrow \varphi$ is derivable in SŁ.

A. Adamson and R. Giles. A Game-Based Formal System for $Ł_\infty$. *Studia Logica* 1(38) (1979), 49–73.

Sequents

A **sequent** is an ordered pair of finite multisets of formulas Γ and Δ, written $\Gamma \Rightarrow \Delta$ (essentially, a *dialogue state*).

The following **sequent rules** represent elements of a strategy:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}$$

Let SŁ be the **sequent calculus** consisting of these rules plus

$$\frac{}{\Gamma, \underbrace{\bot, \ldots, \bot}_{n}, \Delta \Rightarrow \Delta, \varphi_1, \ldots, \varphi_n} \qquad \frac{\Gamma, \varphi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

Theorem (Adamson and Giles)

φ is Ł-valid iff $\Rightarrow \varphi$ is derivable in SŁ.

A. Adamson and R. Giles. A Game-Based Formal System for $Ł_\infty$.
Studia Logica 1(38) (1979), 49–73.

Hypersequents

A **hypersequent** $\mathcal{G}$ is a finite multiset of sequents, written

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

(essentially, a *state disjunction*).

A. Avron. A constructive analysis of RM.
Journal of Symbolic Logic 52(4) (1987), 939–951.

Disjunctive Strategies as Proofs

Similarly to Adamson and Giles, we have **implication rules**

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}$$

We also need **duplication rules**

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \qquad \frac{\mathcal{G} \quad \ldots \quad \mathcal{G}}{\mathcal{G}}$$

Notice: a disjunctive strategy for $[\Gamma \parallel \Delta]$ "is" a proof of $\Gamma \Rightarrow \Delta$ from atomic hypersequents using the implication and duplication rules.

Disjunctive Strategies as Proofs

Similarly to Adamson and Giles, we have **implication rules**

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}$$

We also need **duplication rules**

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \qquad \frac{\mathcal{G} \quad \ldots \quad \mathcal{G}}{\mathcal{G}}$$

Notice: a disjunctive strategy for $[\Gamma \parallel \Delta]$ "is" a proof of $\Gamma \Rightarrow \Delta$ from atomic hypersequents using the implication and duplication rules.

Disjunctive Strategies as Proofs

Similarly to Adamson and Giles, we have **implication rules**

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}$$

We also need **duplication rules**

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \qquad \frac{\mathcal{G} \quad \ldots \quad \mathcal{G}}{\mathcal{G}}$$

Notice: a disjunctive strategy for $[\Gamma \parallel \Delta]$ “is” a proof of $\Gamma \Rightarrow \Delta$ from atomic hypersequents using the implication and duplication rules.

Disjunctive Winning Strategies as Proofs

Theorem (Fermüller and Metcalfe)

The following are equivalent:

1. *There is a proof of* $\Gamma \Rightarrow \Delta$ *from winning atomic hypersequents using the implication and duplication rules.*
2. *There exists a disjunctive winning strategy for me for* $\mathbf{G}([\Gamma \parallel \Delta], \rho)$ *for any consistent regulation* ρ*.*

The Hypersequent Calculus GŁ

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\textsc{id}) \qquad \frac{}{\mathcal{G} \mid \Rightarrow}\ (\textsc{emp}) \qquad \frac{}{\mathcal{G} \mid \bot \Rightarrow \varphi}\ (\bot\Rightarrow)$$

Structural Rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ew}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ec}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\textsc{wl})$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\textsc{split}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{mix})$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_{Ł} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{Ł}$$

G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.

The Hypersequent Calculus GŁ

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\textsc{id}) \qquad \frac{}{\mathcal{G} \mid \Rightarrow}\ (\textsc{emp}) \qquad \frac{}{\mathcal{G} \mid \bot \Rightarrow \varphi}\ (\bot\Rightarrow)$$

Structural Rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ew}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ec}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\textsc{wl})$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\textsc{split}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{mix})$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_{Ł} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{Ł}$$

G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.

The Hypersequent Calculus GŁ

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\textsc{id}) \qquad \frac{}{\mathcal{G} \mid \Rightarrow}\ (\textsc{emp}) \qquad \frac{}{\mathcal{G} \mid \bot \Rightarrow \varphi}\ (\bot\Rightarrow)$$

Structural Rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ew}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\textsc{ec}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\textsc{wl})$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\textsc{split}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\textsc{mix})$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_{Ł} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{Ł}$$

G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.

The Hypersequent Calculus GŁ

Axioms

$$\frac{}{\mathcal{G} \mid \varphi \Rightarrow \varphi}\ (\text{ID}) \qquad \frac{}{\mathcal{G} \mid \Rightarrow}\ (\text{EMP}) \qquad \frac{}{\mathcal{G} \mid \bot \Rightarrow \varphi}\ (\bot\Rightarrow)$$

Structural Rules:

$$\frac{\mathcal{G}}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\text{EW}) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (\text{EC}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta}\ (\text{WL})$$

$$\frac{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}\ (\text{SPLIT}) \qquad \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \quad \mathcal{G} \mid \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ (\text{MIX})$$

Logical Rules

$$\frac{\mathcal{G} \mid \Gamma, \psi \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow)_{Ł} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)_{Ł}$$

G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p, p \Rightarrow p, q}\,(\text{WL})}{q, q \to p \Rightarrow p, p \to q}\,(\Rightarrow\to)_{\text{Ł}}}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p, p \Rightarrow p, q}\,(\text{WL})}{q, q \to p \Rightarrow p, p \to q}\,(\Rightarrow\to)_{\text{Ł}}}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p, p \Rightarrow p, q}\,(\text{WL})}{q, q \to p \Rightarrow p, p \to q}\,(\Rightarrow\to)_{\text{Ł}}}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\textsc{id}) \quad \dfrac{}{p \Rightarrow p}\,(\textsc{id})}{q, p \Rightarrow p, q}\,(\textsc{mix})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\textsc{id}) \quad \dfrac{}{p \Rightarrow p}\,(\textsc{id})}{q, p \Rightarrow p, q}\,(\textsc{mix})}{q, q \to p, p \Rightarrow p, q}\,(\textsc{wl})}{\dfrac{q, q \to p \Rightarrow p, p \to q}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}}\,(\Rightarrow\to)_{\text{Ł}}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p, p \Rightarrow p, q}\,(\text{WL})}{\dfrac{q, q \to p \Rightarrow p, p \to q}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}}\,(\Rightarrow\to)_{\text{Ł}}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\mathrm{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\mathrm{ID})}{q, p \Rightarrow p, q}\,(\mathrm{MIX})}{q, q \rightarrow p \Rightarrow p}\,(\rightarrow\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\mathrm{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\mathrm{ID})}{q, p \Rightarrow p, q}\,(\mathrm{MIX})}{q, q \rightarrow p, p \Rightarrow p, q}\,(\mathrm{WL})}{\dfrac{q, q \rightarrow p \Rightarrow p, p \rightarrow q}{(p \rightarrow q) \rightarrow q, q \rightarrow p \Rightarrow p}\,(\rightarrow\Rightarrow)_{\text{Ł}}}\,(\Rightarrow\rightarrow)_{\text{Ł}}$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \qquad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p, p \Rightarrow p, q}\,(\text{WL})}{q, q \to p \Rightarrow p, p \to q}\,(\Rightarrow\to)_{\text{Ł}}}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \qquad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p, p \Rightarrow p, q}\,(\text{WL})}{q, q \to p \Rightarrow p, p \to q}\,(\Rightarrow\to)_{\text{Ł}}}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}
$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}} \quad \dfrac{\dfrac{\dfrac{}{q \Rightarrow q}\,(\text{ID}) \quad \dfrac{}{p \Rightarrow p}\,(\text{ID})}{q, p \Rightarrow p, q}\,(\text{MIX})}{q, q \to p, p \Rightarrow p, q}\,(\text{WL})}{q, q \to p \Rightarrow p, p \to q}\,(\Rightarrow\to)_{\text{Ł}}}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{\text{Ł}}$$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{q \Rightarrow q}\,(\textsc{id}) \qquad \dfrac{}{p \Rightarrow p}\,(\textsc{id})
    }{q, p \Rightarrow p, q}\,(\textsc{mix})
  }{q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{Ł}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{}{q \Rightarrow q}\,(\textsc{id}) \qquad \dfrac{}{p \Rightarrow p}\,(\textsc{id})
    }{q, p \Rightarrow p, q}\,(\textsc{mix})
  }{q, q \to p, p \Rightarrow p, q}\,(\textsc{wl})
}{
  \dfrac{q, q \to p \Rightarrow p, p \to q}{(p \to q) \to q, q \to p \Rightarrow p}\,(\to\Rightarrow)_{Ł}
}\,(\Rightarrow\to)_{Ł}
$$

Variants of the Game

We can have different

- **winning conditions** (e.g., for classical logic, *n*-valued logics)
- **dialogue rules** (e.g., for abelian logic, Chang's logic)
- **structures** (e.g., for Gödel logic, product logic).

Variants of the Game

We can have different

- **winning conditions** (e.g., for classical logic, *n*-valued logics)
- **dialogue rules** (e.g., for abelian logic, Chang's logic)
- **structures** (e.g., for Gödel logic, product logic).

Variants of the Game

We can have different

- **winning conditions** (e.g., for classical logic, *n*-valued logics)
- **dialogue rules** (e.g., for abelian logic, Chang's logic)
- **structures** (e.g., for Gödel logic, product logic).

Some Remarks on First-Order Łukasiewicz Logic

- **First-order Łukasiewicz logic** – where $\forall$ and $\exists$ are interpreted by infs and sups, respectively – is not recursively enumerable.
- Let GŁ$\forall$ be GŁ extended with standard quantifier rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)\varphi(x), \Delta} \quad \frac{\mathcal{G} \mid \Gamma, \varphi(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)\varphi(x) \Rightarrow \Delta} \quad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)\varphi(x), \Delta} \quad \frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}$$

where a is a free variable not occurring in the premises.

- GŁ$\forall$ extended with a cut rule is complete with respect to algebraic semantics but does not admit cut-elimination.
- However, a first-order formula φ is Ł-valid iff $\bot \Rightarrow \overbrace{\varphi, \ldots, \varphi}^{n}$ is derivable in GŁ$\forall$ for all $n \geq 1$.

M. Baaz and G. Metcalfe. Herbrand's Theorem, Skolemization, and Proof Systems for Łukasiewicz Logic. *Journal of Logic and Computation* 20 (2010), 35–54.

Some Remarks on First-Order Łukasiewicz Logic

- **First-order Łukasiewicz logic** – where $\forall$ and $\exists$ are interpreted by infs and sups, respectively – is not recursively enumerable.
- Let GŁ$\forall$ be GŁ extended with standard quantifier rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)\varphi(x) \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}$$

where a is a free variable not occurring in the premises.

- GŁ$\forall$ extended with a cut rule is complete with respect to algebraic semantics but does not admit cut-elimination.
- However, a first-order formula φ is Ł-valid iff $\bot \Rightarrow \overbrace{\varphi, \ldots, \varphi}^{n}$ is derivable in GŁ$\forall$ for all $n \geq 1$.

M. Baaz and G. Metcalfe. Herbrand's Theorem, Skolemization, and Proof Systems for Łukasiewicz Logic. *Journal of Logic and Computation* 20 (2010), 35–54.

Some Remarks on First-Order Łukasiewicz Logic

- **First-order Łukasiewicz logic** – where $\forall$ and $\exists$ are interpreted by infs and sups, respectively – is not recursively enumerable.
- Let GŁ$\forall$ be GŁ extended with standard quantifier rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)\varphi(x) \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}$$

where a is a free variable not occurring in the premises.

- GŁ$\forall$ extended with a cut rule is complete with respect to algebraic semantics but does not admit cut-elimination.
- However, a first-order formula φ is Ł-valid iff $\bot \Rightarrow \overbrace{\varphi, \ldots, \varphi}^{n}$ is derivable in GŁ$\forall$ for all $n \geq 1$.

M. Baaz and G. Metcalfe. Herbrand's Theorem, Skolemization, and Proof Systems for Łukasiewicz Logic. *Journal of Logic and Computation* 20 (2010), 35–54.

Some Remarks on First-Order Łukasiewicz Logic

- **First-order Łukasiewicz logic** – where $\forall$ and $\exists$ are interpreted by infs and sups, respectively – is not recursively enumerable.
- Let GŁ$\forall$ be GŁ extended with standard quantifier rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)\varphi(x) \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}$$

where a is a free variable not occurring in the premises.

- GŁ$\forall$ extended with a cut rule is complete with respect to algebraic semantics but does not admit cut-elimination.
- However, a first-order formula φ is Ł-valid iff $\bot \Rightarrow \overbrace{\varphi, \ldots, \varphi}^{n}$ is derivable in GŁ$\forall$ for all $n \geq 1$.

M. Baaz and G. Metcalfe. Herbrand's Theorem, Skolemization, and Proof Systems for Łukasiewicz Logic. *Journal of Logic and Computation* 20 (2010), 35–54.

Some Remarks on First-Order Łukasiewicz Logic

- **First-order Łukasiewicz logic** – where $\forall$ and $\exists$ are interpreted by infs and sups, respectively – is not recursively enumerable.
- Let GŁ$\forall$ be GŁ extended with standard quantifier rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(a), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\forall x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(t) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\forall x)\varphi(x) \Rightarrow \Delta} \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi(t), \Delta}{\mathcal{G} \mid \Gamma \Rightarrow (\exists x)\varphi(x), \Delta} \qquad \frac{\mathcal{G} \mid \Gamma, \varphi(a) \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, (\exists x)\varphi(x) \Rightarrow \Delta}$$

where a is a free variable not occurring in the premises.

- GŁ$\forall$ extended with a cut rule is complete with respect to algebraic semantics but does not admit cut-elimination.
- However, a first-order formula φ is Ł-valid iff $\bot \Rightarrow \overbrace{\varphi, \ldots, \varphi}^{n}$ is derivable in GŁ$\forall$ for all $n \geq 1$.

M. Baaz and G. Metcalfe. Herbrand's Theorem, Skolemization, and Proof Systems for Łukasiewicz Logic. *Journal of Logic and Computation* 20 (2010), 35–54.

Concluding Remarks

- **Giles's game** provides a natural home for the proof theory of **Łukasiewicz logic**.
- Proofs in a **hypersequent calculus** correspond to **disjunctive strategies** for the game.
- Disjunctive strategies might provide a more flexible framework for investigating relationships between **games** and **proof theory**.

Concluding Remarks

- **Giles's game** provides a natural home for the proof theory of **Łukasiewicz logic.**
- Proofs in a **hypersequent calculus** correspond to **disjunctive strategies** for the game.
- Disjunctive strategies might provide a more flexible framework for investigating relationships between **games** and **proof theory.**

Concluding Remarks

- **Giles's game** provides a natural home for the proof theory of **Łukasiewicz logic**.
- Proofs in a **hypersequent calculus** correspond to **disjunctive strategies** for the game.
- Disjunctive strategies might provide a more flexible framework for investigating relationships between **games** and **proof theory**.