Implications as rules

In defence of proof-theoretic semantics

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Two dogmas of standard semantics

D. 1 The categorical is conceptually prior to the hypothetical
— the priority of the categorical over the hypothetical—

D. 2 Consequence is defined as the transmission of the basic categorical concept from the premisses to the conclusion
— the transmission view of consequence —

Model-theoretic consequence

$$A \models B \; := \; (\forall \mathfrak{M})(\mathfrak{M} \models A \;\Rightarrow\; \mathfrak{M} \models B\,)$$

"Every model of the premisses is a model of the conclusion"

Constructive consequence

$$A \models B \; := (\forall \mathcal{C})(\mathcal{C} \models A \;\Rightarrow\; f(\mathcal{C}) \models B)$$

(BHK, Lorenzen's admissibility interpretation of implication)

We use truth-makers (constructions, proofs) and constructive transformations.

Material implication

$$\mathfrak{M} \models A \rightarrow B \ := \ (\mathfrak{M} \models A \ \Rightarrow \ \mathfrak{M} \models B)$$

Constructive material implication

$$\mathfrak{S} \models A \rightarrow B \ := (\forall \mathcal{C})(\mathcal{C}, \mathfrak{S} \models A \ \Rightarrow \ f(\mathcal{C}), \mathfrak{S} \models B)$$

There is a quantifier already in the material case. The transmission view already governs material implication.

Although never formulated that way, the critique of the transmission view has fostered dialogical / game-theoretical semantics.

Critique of the transmission view

- Global view of deductive reasoning: Cannot deal with local (partial) meaning and non-wellfounded phenomena
- ‘Non-definiteness’ of notion of proof or construction: Lack of proper meaning explanation
  - Iteration of implication in Lorenzen’s admissibility concept (improper ‘meta-calculi’)
  - Realizability: Not decidable of whether e is an index with certain properties
  - “Impredicativity of implication”
    $f : (A \to A) \to A \qquad \lambda x.fx$ as argument of f
  - Beyond monotone inductive definitions

Counterargument: Validity can be established

- By giving a 'derivation' in a meta-calculus
- By providing a construction according to the BHK explanation
- By giving a realizing index

The only problem is completeness.

But is this a problem?

The essential argument is an epistemological one: A speaker cannot grasp the meaning when it is explained according to the transmission view. Therefore a 'combinatorial' way of explaining meaning is needed.

Lorenz: The notion of proposition remains unexplained otherwise.

Dialogical logic and ‘definiteness’

‘Non-definiteness’ of standard constructive semantics has been used as an argument in favour of dialogical logic.

- Plays as the level of meaning explanations, leading to a constructive notion of ‘proposition’.
- Strategies correspond to the level of proofs.

Important is not so much the difference between plays and strategies, but the fact that even at the level of strategies, we have a strict codification of constructions. (Some game-theoretic semanticists dispute this.)

Unlike proofs, the concept of strategy is not iterated.

In defence of proof-theoretic semantics

- The problem is implication
- We can do without the transmission view
- Implications as rules
- Only the applicative behaviour if implication is relevant
- Implication is treated separately from the other logical constants — but not in the intuitionistic/constructive sense

Left-iterated implications

Observation: Iteration of implication only relevant on the left side:

- $A \rightarrow (B \rightarrow C)$ is $A \wedge B \rightarrow C$
- without conjunction, written in sequent-style:
  $A \rightarrow (B \rightarrow C)$ is $A, B \rightarrow C$

From a sequent-style perspective, this means that implications are only relevant in antecedent position (at least in a purely implicational system)

$(A \rightarrow B) \rightarrow (C \rightarrow (D \rightarrow E))$ becomes $(A \rightarrow B), C, D \rightarrow E$
or $(A \rightarrow B), C, D \vdash E$

Proposal: Implications as rules

Claim: Implication is different from other constants.

- It is to be viewed as a rule, which operates essentially on the left (assumption) side.
- Symmetry / harmony does not apply to implication.
- Rather, implications-as-rules are presupposed for the dealing with harmony principles.
- Conclusion: The (purported) arguments against proof-theoretic semantics are no longer valid.

This is a defence of proof-theoretic semantics, not an argument against game-theoretic semantics.

(In fact, our rule-based reading of implications gives rise to a certain game-theoretic treatment.)

Left-iterated implications as rules

$Rule ::= Atom \mid (Rule, \ldots, Rule \Rightarrow Atom)$

Intended meaning of $((\Gamma_1 \Rightarrow A_1), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow B)$:
If each A_i has been derived from Γ_i, respectively, then we may pass over to B.

$$\frac{\begin{array}{ccc}\Gamma_1 & & \Gamma_n \\ A_1 & \ldots & A_n\end{array}}{B}$$

In a sequent-style framework:

$$\frac{\Delta, \Gamma_1 \vdash A_1 \quad \ldots \quad \Delta, \Gamma_n \vdash A_n}{\Delta \vdash B}$$

Schema for rule application

$$\frac{\Delta, \Gamma_1 \vdash A_1 \quad \ldots \quad \Delta, \Gamma_n \vdash A_n}{\Delta, ((\Gamma_1 \Rightarrow A_1), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow B) \vdash B}$$

This generalizes the schema

$$\frac{\Gamma \vdash A}{\Gamma, (A \rightarrow B) \vdash B}$$

This is not a definition of implication based on some sort of harmony, but gives implication an elementary meaning.

Right-iteration as abbreviation

$\Gamma \vdash A \Rightarrow (B \Rightarrow C)$ understood as $\Gamma, A, B \vdash C$

i.e., we are dealing with list structures.

Initial sequents: $R \vdash R$

This means: $R, (R)_1 \vdash (R)_2$

For example: $(\Gamma \Rightarrow A), \Gamma \vdash A$

This involves the reading of implications as rules.

Not simply: Right and left side are identical.

Justification of cut

$$\frac{\Gamma \vdash R \quad \Delta, R \vdash C}{\Gamma, \Delta \vdash C}$$

Example:

$$\frac{\Gamma, A \vdash B \quad \Delta, (A \Rightarrow B) \vdash C}{\Gamma, \Delta \vdash C}$$

Justification: The left premiss eliminates the application of $A \Rightarrow B$ in the right premiss.

This yields an elementary Frege calculus.

Implications-as-rules from the database perspective: *resolution*

Suppose the implication $A \rightarrow B$ is available in our database.

Then the goal B can be reduced to the goal A.

More generally: Given a database (or logic program)

$$\begin{cases} B \leftarrow A_1 \\ \quad \vdots \\ B \leftarrow A_n \end{cases}$$

then the goal B can be reduced to any of the goals A_i.

This reduction is called ‘resolution’.

Reasoning with respect to a database of implications means reading them as rules.

Generalization: Clausal definitions and common content

Given a clausal definition

$$\mathbb{D} \left\{ \begin{array}{lll} A & \text{:-} & \Delta_1 \\ & \vdots & \\ A & \text{:-} & \Delta_n \end{array} \right.$$

then A is intended to express the common content of $\Delta_1, \ldots, \Delta_n$:

For all R: $A \vdash R$ iff $\Delta_1 \vdash R, \ \ldots \ , \Delta_n \vdash R$

This gives the usual right- and left rules:

$$\frac{\Gamma \vdash \Delta_i}{\Gamma \vdash A} \qquad\qquad \frac{\Gamma, \Delta_1 \vdash C \quad \ldots \quad \Gamma, \Delta_n \vdash C}{\Gamma, A \vdash C}$$

At this level we have symmetry / harmony !

Result

Implication has a non-symmetric primordial meaning, other constants are symmetrically defined.

We can define $A \rightarrow B$ in terms of the rule $A \Rightarrow B$.

This allows us to interpret a nested implicational formula such as $(A \rightarrow B) \vee (C \rightarrow D)$.

Remarks on cut

Better option in the spirit of the rule-interpretation:

Use a weaker background logic, based only on rule application

$$\frac{\Delta \vdash A}{\Delta, (A \Rightarrow B) \vdash B}$$

and its generalization

$$\frac{\Delta, \Gamma_1 \vdash A_1 \quad \ldots \quad \Delta, \Gamma_n \vdash A_n}{\Delta, ((\Gamma_1 \Rightarrow A_1), \ldots, (\Gamma_n \Rightarrow A_n) \Rightarrow B) \vdash B}$$

without having cut as primitive.

Summary

Our case for proof-theoretic semantics:

- By giving implication an elementary combinatorial meaning (implications-as-rules) we avoid the problems that have led Lorenzen, Lorenz and (some of their) followers to abandon proof-theoretic in favour of dialogical semantics
- Symmetry / harmony comes into play only after implications-alias-rules are already available
- The critique of the transmission view of consequence speaks against certain types of proof-theoretic semantics (BHK, Lorenzen, Dummett-Prawitz), but not against proof-theoretic semantics as such

This is no case against game-theoretical semantics !

Personally, as a proof-theoretic semanticist, I favour Lorenzen I over Lorenzen II.

References

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Thomas Piecha: Implications as rules in dialogues. Next talk at this conference.

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