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The Categorical and the Hypothetical: A critique of some fundamental assumptions of standard semantics

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Introduction

By standard semantics we mean classical model-theoretic semantics in the tradition of Tarski as well as intuitionistic and proof-theoretic semantics in the BHK (Brouwer-Heyting-Kolmogorov) tradition and in the proof-theoretic tradition of Dummett, Lorenzen, Martin-Löf and Prawitz[1]. Both the classical and the intuitionistic / proof-theoretic traditions rest on certain fundamental assumptions which will be criticized in this paper and will be opposed to an alternative variant of proof-theoretic semantics based on different ideas. The three fundamental assumptions criticized are the following:

I. The categorical is conceptually prior to the hypothetical — *the priority of the categorical over the hypothetical*

II. Consequence is defined as the transmission of a basic categorical concept from the premisses to the conclusion — *the transmission view of consequence*

III. Consequence means the same as correctness of inference — *the identification of consequence and correctness of inference.*

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[1]For the term and the notion 'proof-theoretic semantics' see [24].

As an alternative view, we propose a variant of proof-theoretic semantics according to which

I*. The hypothetical is conceptually prior to the categorical

II*. Consequence is a primitive concept which is not explained as the transmission of categorical truth or canonical provability

III*. Consequence is a basic semantical notion which does not necessarily, and in fact not always, coincide with correctness of inference.

1. Standard notions of consequence: The first two dogmas

It is important to distinguish between *formal* and *material consequence*. Formal consequence is logical consequence, which is material consequence for every 'material' base. In the model-theoretic case this material base is a structure or interpretation, which says what the domain of reasoning is, and which meaning the non-logical signs have. In the proof-theoretic case it is the notion of an atomic proof system which generates the atomic truths. In the general intuitionistic case it is the notion of an elementary construction that validates atomic sentences. If we denote material consequence with respect to a material base $\mathfrak{B}$ by $\models_{\mathfrak{B}}$, then logical or formal consequence, denoted by $\models$, would be defined by

$$A \models B \quad =_{def} \quad (\forall \mathfrak{B})\ A \models_{\mathfrak{B}} B\ . \tag{1}$$

Here and in the following, with a few exceptions, we assume for simplicity that consequences have only a single sentence in the antecedent[2]. Therefore, the basic concept to be explained is that of material consequence '$\models_{\mathfrak{B}}$'. A (material) statement of the form $A \models_{\mathfrak{B}} B$ will also be called *consequence statement* or *hypothetical judgement*. The difference between classical model-theoretic semantics and intuitionistic or proof-theoretic semantics is a difference at the level of material consequence.

In classical model-theoretic semantics, where $\mathfrak{B}$ is a structure $\mathfrak{M}$, material consequence $A \models_{\mathfrak{M}} B$ is explained as

$$A \models_{\mathfrak{M}} B \quad =_{def} \quad (\models_{\mathfrak{M}} A \quad \Rightarrow \quad \models_{\mathfrak{M}} B)\ , \tag{2}$$

where " $\Rightarrow$ " denotes ordinary (material) implication and "$\models_{\mathfrak{M}} A$" means that A holds in $\mathfrak{M}$, i.e. that $\mathfrak{M}$ is a model of A. "$\models_{\mathfrak{M}} A$" is defined in the usual way along the lines of Tarski. If we consider "$\models_{\mathfrak{M}} A$" to be a categorical concept (the truth of A with respect to a structure) and $A \models_{\mathfrak{M}} B$ a hypothetical concept (the hypothetical

[2]This is just a matter of abbreviation, dispensing us with considering sets or multisets or sequences on the left of the consequence sign. The necessary adjustments needed can easily be carried out by the reader.

validity of B given the assumption A [again with respect to a structure]), then (2) defines the hypothetical concept of (material) consequence in terms of the categorical concept of truth [with respect to a structure $\mathfrak{M}$]. Therefore, with respect to the order of explanation, the categorical concept of truth comes first, and the hypothetical concept of consequence comes second. Furthermore, the way the hypothetical is defined in terms of the categorical can be expressed as the transmission of the categorical (i.e., truth) from the antecedent (= left side) of the consequence statement to its consequent (= right side). This is exactly what the implication " $\Rightarrow$ " says: If the left side is true then so is the right side. Therefore, in the classical model-theoretic case, the first two dogmas: (I) the priority of the categorical over the hypothetical, and (II) the transmission view, figure prominently in the definition of consequence.

In intuitionistic and proof-theoretic semantics — in the following subsumed under the heading *constructive* semantics — the notion of truth is replaced with the notion of construction or proof. There are various forms of this semantics, from different forms of the BHK interpretation of logical constants over Kleene's realizability interpretation, the Curry-Howard interpretation, the meaning explanations in Martin-Löf's type theory, and Lorenzen's admissibility interpretation to Dummett's and Prawitz's proof-theoretic notions of validity. In all these variants, the notion of a hypothetical judgement is explained in terms of a constructive procedure that transforms a construction of the antecedent into a construction of the consequent. If we denote by $\mathcal{C} \models_{\mathfrak{S}} A$ the fact that $\mathcal{C}$ is a construction or proof of A with respect to the atomic base $\mathfrak{S}$, we can represent their notion of material consequence (i.e., consequence with respect to $\mathfrak{S}$) by the following definition:

$$A \models_{\mathfrak{S}} B \;=_{def}\; (\exists f)(\forall \mathcal{C})(\mathcal{C} \models_{\mathfrak{S}} A \;\Rightarrow\; f(\mathcal{C}) \models_{\mathfrak{S}} B)\ , \tag{3}$$

in words: B follows from A (w.r.t. $\mathfrak{S}$) if we can find a (constructive) transformation f that associates with every construction or proof $\mathcal{C}$ of A a construction or proof $f(\mathcal{C})$ of B.

Compared to the classical (model-theoretic) case this definition is more involved: We do not merely use (material) implication " $\Rightarrow$ ", but require in addition the existence of a constructive transformation (function) that delivers a construction of the consequent given an arbitrary construction of the antecedent of the hypothetical judgement. This difference is, of course, crucial, as it makes the intuitionistic or proof-theoretic notion a constructive one, compared to the classical case, which just relies on the given intuitive meaning of implication " $\Rightarrow$ " without caring about how the relationship between antecedent and consequent of a hypothetical judgement is established. This also gives the constructive notion of consequence a natural epistemological rendering, which is a basic ingredient of intuitionism.

However, as far as our first two dogmas are concerned, there is not so much difference to the model-theoretic approach. In constructive semantics the categorical concept $\mathcal{C} \models_{\mathfrak{S}} A$ comes first, and the hypothetical concept $A \models_{\mathfrak{S}} B$ is, according to (3), explained as the transmission of the categorical concept from the antecedent A to the consequent B. Definition (3) uses the notion of construction as an additional parameter, but this does not change the general picture. Both the classical and the constructivist approaches to semantics rely on the *priority of the categorical over the hypothetical* and on the *transmission view of consequence*, and as these two conceptions of semantics may be called "standard semantics", it is justified to speak of *two dogmas of standard semantics* [20].

2. Consequence and correctness of inference: The third dogma

What is the reason for starting with the categorical concept (truth / proof) and define the hypothetical concept of consequence as the transmission of the categorical one? This is not too difficult to see. If by reasoning we understand a chain of inferences that proceeds stepwise from premisses to conclusion, then, so one might argue, our notion of consequence should be such that all and only correct inferences are rendered correct. Now an inference is correct if it leads from correct premisses to a correct conclusion. The easiest way of incorporating this into a notion of consequence is to *define* a correct consequence statement as something that leads from a correct statement to a correct statement. In fact, standard proof-theoretic semantics comes close to such a definition. Now in the classical case, we do not define consequence in terms of inference. However, as soon as we deal with inference, we (at least implicitly) identify *consequence* and the *correctness of inference* in the sense that both notions have the same definiens.

If we define the correctness of an inference

$$\frac{A}{B}$$

classically, we require that by passing from A to B we stay in the realm of truth, i.e., we require what is expressed by the right side (the definiens) of (2). If we define the correctness of this inference constructively, we require that by passing from A to B we are applying a constructive transformation or function that gives us a construction of B out of a construction of A, i.e., we require what is expressed by the right side (the definiens) of (3). Therefore, in both the classical and the constructive case consequence is defined in exactly the same way in which the correctness of an inference is defined. What differs between the classical and the constructive case is the order of explanation. In the classical case the semantical notion of consequence comes first, and the correctness of inference is explained by reference to this semantical notion: An inference is correct, if it follows a correct consequence. In the constructive case the epistemological notion of inference comes first, and a correct consequence is explained by reference to this epistemological notion: A consequence is correct, if it represents a correct inference step. In both cases these two notions coincide. This is not surprizing at all. It simply means that for inferences to be drawn in a correct way, it is necessary and sufficient that they accord to the semantics of the expressions involved.

We call this the *third dogma of standard semantics*: The identification of consequence with the correctness of inference. Note that we do not speak of the identification of consequence with correct inference, as this would conflate a semantical notion (consequence) with an epistemic act (inference), but of consequence with the correct*ness* of inference, the latter being a semantical property of inferences.

Altogether, without mentioning the atomic base or structure, and without mentioning constructive transformations, the basic tenet of standard semantics can be paraphrased as

$$A \models B \quad := \quad (\models A \;\Rightarrow\; \models B) \quad =: \quad (\text{the inference } \frac{A}{B} \text{ is correct})$$

where in the middle we have the joint definiens of the notions on the left and on the right side. As we do not dispute the definition on the right

$$(\text{the inference } \frac{A}{B} \text{ is correct}) \quad := \quad (\models A \;\Rightarrow\; \models B)\ ,$$

for us the critical fundamental assumption of standard semantics, which will be questioned in the rest of this paper, is

$$A \models B \quad := \quad (\models A \;\Rightarrow\; \models B)\ . \tag{4}$$

3. Consequence as a basic semantical concept

The alternative conception propagated here is to break up the identification (4) and keep the concept of consequence ($A \models B$) separate from the concept of correctness of inference or transmission of truth ($\models A \;\Rightarrow\; \models B$). Hypothetical consequence ($A \models B$) will be defined directly without reference to categorical truth or validity $\models A$, so the hypothetical concept will be prior to the categorical concept, which means that the first dogma of standard semantics is given up. In fact, it will be reversed, as the concept of categorical truth or validity $\models A$ will become a limiting case of the hypothetical concept $A \models B$, viz. the case where the antecedent is lacking. The second dogma, the transmission view of consequence, is given up as well. The categorical concept of truth or validity is no longer available prior to consequence, and there will be no other concept whose transmission would define consequence. The notion of transmission or transformation does not play a role in the definition of consequence any more.

Giving up (4) as a definitional equation and defining consequence independently, means that we may ask whether what in (4) is forced by definition is in fact valid, i.e., whether the biconditional

$$A \models B \quad \Leftrightarrow \quad (\models A \;\Rightarrow\; \models B) \tag{5}$$

holds. This can, of course, not be answered in general, but will depend on our domain of reasoning. So instead of (5) we might ask whether with respect to a specific material base $\mathfrak{B}$

$$A \models_{\mathfrak{B}} B \quad \Leftrightarrow \quad (\models_{\mathfrak{B}} A \;\Rightarrow\; \models_{\mathfrak{B}} B) \tag{6}$$

holds.

Now what should a material base $\mathfrak{B}$ look like for (6) *not* to hold? Should we not expect our base always be such as to validate (6), since the most natural *expectation* is that consequence ensure correct inference? We shall give counterexamples later on. First we have to elucidate what $A \models_{\mathfrak{B}} B$ should mean.

We define $A \models_{\mathfrak{B}} B$ by means of certain rules or inference principles which govern the meaning of the terms involved. In this sense the definition of the semantical concept $A \models_{\mathfrak{B}} B$ is inferential. However, these semantical rules, also called "rules of consequence"[3] are to be distinguished from inferential principles justified as correct $(\models_{\mathfrak{B}} A \;\Rightarrow\; \models_{\mathfrak{B}} B)$. In particular, the notion of correctness of inference is *not* presupposed in the inferential definition of $A \models_{\mathfrak{B}} B$, which means that the third dogma is given up as well.

3.1. The 'material' base generalized: Clausal definitions

Normally the material base of consequence deals with the atomic case. In classical model-theoretic semantics this would be a structure which determines which atomic sentences are valid. In constructive semantics it would be elementary constructions or proofs for atomic sentences, e.g. atomic systems in Prawitz's [14] sense. Starting from atomic sentences, logically complex sentences are built up by means of logical operators. The complex sentences are then valid or invalid according to certain meaning explanations (truth-functional or constructive, respectively) for the logical operators.

This presupposes that there is a clearcut distinction between the atomic and the complex sentences. In standard semantics this is given by the syntax of sentences. Certain sentences are defined as atomic, whereas others are logically complex, simply because they contain logical operators. The distinction between atomic and non-atomic is 'hardwired' into the formalism, which means that the material base, which may vary, only concerns the non-logical aspects.

This may be a plausible way of proceeding, if our ultimate goal is the elucidation of logical (i.e., formal) consequence. However, there is no reason in principle why the composition by means of logical operators should be treated differently from any other sort of composition. Even if logical composition represents a particularly important case, this can be treated as a special case amongst other cases. Proceeding that way gives us a much greater generality and a wider range of applications. The laws which

[3]This term has been proposed by Luca Tranchini.

explain the meanings of the logical operators can be embedded in a more general framework of rule-based meaning explanations. So we are not just considering generalized rules for logical constants as done, for example in [15, 16], where the *logical* meaning explanations are generalized, but arbitrary definitions. The case of logical constants or of generalized logical constants can then be obtained by particular definitions.

As the general format for definitions we choose clauses of the form

$$a \;:=\; B \tag{7}$$

where a is an object to be defined and the B is a defining condition for this object. As in logic programming, we also speak of the head a and the body B of the clause. If we want to draw a distinction between atoms and complex entities, the definiendum a is always atomic, whereas a defining condition B can be complex. What complexity here means, depends on the context considered. In the simplest case, a defining condition B is just a list of atoms, so that (7) takes the form

$$a \;:=\; b_1, \ldots, b_n \;. \tag{8}$$

More generally, the condition B may, in addition to the comma (which is a structural conjunction), contain some sort of structural implication '$\rightarrow$', so that one can have a clause such as, e.g.,

$$a \;:=\; b, (c \rightarrow d), (e \rightarrow f) \tag{9}$$

Further generalizations would allow for (structural) quantifiers in the body etc., or even infinite sets as bodies (corresponding to infinitary [structural] conjunction, see [4]). However, most important is the syntax of the objects defined. In all practical applications we would use variables occurring in defined (and defining) objects, giving them a term or formula structure. This leads to definitions syntactically corresponding to logic programs.

What is important already in its simplest form is that there may be more than one defining clause for an object a. This is a way to express disjunctive meaning at the definitional level and is well known from logic programming. It is a feature in which definitions in our sense go beyond explicit definitions, which would only contain a single clause.

We propose the clausal format as it has proven quite powerful. Systems of clauses can be viewed from various perspectives, not only as logic programs but also as inductive definitions. Both the study of logic programs and that of inductive definitions have demonstrated their expressiveness.

It is important to notice that the usual atomic / non-atomic distinction in logic has nothing to do with the atomic / non-atomic distinction between defined objects and defining conditions. If we define logical constants, then everything being defined is atomic in the definitional sense, even if it is complex in the logical sense. This is why we prefer speaking of defined *objects* rather than defined *atoms*.

The idea of considering systems of clauses as basic definitional systems goes back to well before logic programming. In the context of recursive function theory, such systems have been considered by Post [11, 12] and Smullyan [25]. They figure prominently in Lorenzen's operative foundation for logic based on the concept of admissibility [8, 9].

In the following, a *definition* $\mathbb{D}$ is a set of definitional clauses, where we leave open the precise structure of defined objects and defining conditions. It will always be clear from the context which prerequisites are needed.

3.2. Consequence as constituted by definitional principles

Given a definition $\mathbb{D}$, the consequence relation $\models_{\mathbb{D}}$ based on $\mathbb{D}$ is defined by certain rules of consequence which generate expressions of the form $A \models B$. Thus $A \models_{\mathbb{D}} B$ means that an expression ('sequent') of the form $A \models B$ can be generated (i.e., derived) with respect to the definition $\mathbb{D}$. We omit the reference to the definition $\mathbb{D}$, speaking just of $\models$, when the reference is clear from the context, and when it is clear whether we mean the expression $A \models B$ or the statement $A \models_{\mathbb{D}} B$. Besides general structural principles, the inferential definition of $\models_{\mathbb{D}}$ is based on two major rules of consequence: *definitional closure* and *definitional reflection*. Further principles, such as induction rules, might be considered in addition, but will not be discussed here.

The rule of definitional closure is a rule for introducing an object on the right side of a consequence statement. It just expresses the reasoning along a single clause, expressing that given the body of a clause (a definiens), we may proceed to the head (the definiendum). This is the principle on which logic programming is based, with SLD resolution as the evaluation method for backward reasoning. If (7) is a defining clause for a in $\mathbb{D}$, the corresponding rule of definitional closure is

$$\frac{A \models B}{A \models a}\ (aR) \qquad\qquad \textit{definitional closure}$$

The rule of definitional reflection is a kind of inverse of that rule, representing the corresponding left introduction rule. It is a more global principle not referring to a single clause in a definition, but to the definition as a whole. Without considering variables in clauses, it runs as follows: Suppose a is defined by exactly n clauses

$$\begin{cases} a & := & B_1 \\ & \vdots & \\ a & := & B_n \end{cases}$$

then the rule of definitional reflection for a is

$$\frac{B_1 \models A \quad \ldots \quad B_n \models A}{a \models A}\ (aL) \qquad\qquad \textit{definitional reflection}$$

It says that everything that follows from each defining condition of a follows from a itself. Definitional reflection may be looked upon as formally expressing the extremal clause of the inductive definition of a: "Nothing else defines a".[4] If there is only a single defining clause

$$a \ := \ B$$

for a, we obtain the axiom $a \models B$ as a 'direct' inversion of the closure rule. If a is not defined at all, we obtain $a \models A$ for any A, i.e., *ex falso quodlibet*, since the premiss of aL is vacuously fulfilled.[5]

Though looking innocuous, definitional reflection adds considerable deductive power to the evaluation of a clausal definition. In focusing on all clauses of the definition of a and not just on a single defining clause for a, is non-monotonic in the sense that consequence statements can loose their validity if the underlying definition is extended.[6]

In addition to definitional closure and definitional reflection there must be certain structural principles such as initial sequents

$$A \models A$$

and principles governing structural conjunction and structural implication. For structural implication we might use as rules of consequence the principles

$$\frac{A, B \models C}{A \models B \to C}\ (\to R) \qquad \frac{A \models B}{A, B \to C \models C}\ (\to L)$$

which interpret structural implication as a kind of 'rule' that can be *established* $(\to R)$ or *applied* $(\to L)$.[7]

However, the cut rule is not presupposed. This is absolutely crucial. As in the standard sequent calculus, this rule does not convey any meaning to the sentences involved, but serves, if it is admissible at all, to shorten certain derivations[8]. In fact, it is closely related to the idea of correctness of inference. So our definition of consequence

[4]Note that according to our convention to just mention a single object in the antecedent, we have not mentioned any context of the defining conditions B_i in (aL). More precisely, we should have written $\Gamma, B_i \models A$.

[5]However, we may use a metalinguistic framework, in which such a limiting case is not allowed or is handled in a special way. This cannot be discussed here (see [17]).

[6]For further discussions see [6].

[7]The schema $(\to L)$ differs from Gentzen's schema for the introduction of implication in the antecedent, but is particularly suited to our reading of structural implication as a kind of 'rule', see [22, 23].

[8]Yielding, for example, non-elementary (and therefore more than exponential) speed-up in first-order logic, see [10, 26].

with respect to a definition $\mathbb{D}$ corresponds to a cut-free sequent calculus with added definitional principles.[9] By putting a definition $\mathbb{D}$ into action, this system gives the semantics of the objects defined in $\mathbb{D}$ by generating a consequence relation based on $\mathbb{D}$. Derivations of consequence assertions $A \models B$ in the system just sketched are also called *semantical derivations with respect to* $\mathbb{D}$ as they establish the semantical consequence claim $A \models_{\mathbb{D}} B$.[10]

3.3. Defining the logical constants

The logical operators $\wedge$, $\vee$, $\supset$ and $\neg$ can be defined by the following definition:

$$\mathbb{L}\begin{cases} (X\wedge Y) & := & X, Y \\ (X\vee Y) & := & X \\ (X\vee Y) & := & Y \\ (X\supset Y) & := & (X \to Y) \\ \neg X & := & (X \to \bot) \end{cases}$$

where X and Y range over formulas built up from propositional variables by means of $\wedge$, $\vee$, $\supset$ and $\neg$, and where $\bot$ is an undefined object, i.e. an object, for which no definitional clause in $\mathbb{D}$ is given. This definition would have to be extended with rules for the propositional variables representing the material base. If we are only considering formal logic, we would just add the void clauses

$$p := p$$

for every definitional variable p, which express that p is not given any specific meaning.[11] This system is called $\mathbb{LI}$ and yields, by means of closure and reflection, intuitionistic propositional logic.

Our framework is intuitionistic in spirit. In order to obtain classical logic we would have to consider additional reflection principles such as a principle corresponding to classical dilemma in order to make sure that $p\vee\neg p$ becomes a logical law. However, the validity of a formula under a specific valuation can be established on the basis of the definition $\mathbb{L}$ extended with a truth constant with an empty defining condition

$$\top \ :=$$

[9] Such systems have recently been investigated by Brotherston and Simpson [1], albeit in a framework without structural implication.

[10] The idea of definitional reflection and the terms "definitional closure" and "definitional reflection" have been proposed by Hallnäs, who investigated this framework in the context of infinitary clauses [3, 4]. The handling of finitary clauses with variables is due to Hallnäs and Schroeder-Heister [5]. A survey of definitional reflection is under preparation [6].

[11] If we left p just undefined, it would behave like falsum $\bot$. So the fact that a propositional variable p can stand for anything is expressed by a trivial (void) definition for p, which does not assign any *specific* meaning to p.

but without any propositional variables, i.e., with all formulas built up from $\top$ and $\bot$ by means of the propositional connectives. This system is called $\mathbb{LC}$. For the validity of a formula X under the valuation v, we investigate in $\mathbb{LC}$ the closed formula in which every propositional variable p, which is evaluated to *true* under v, is substituted with $\top$, and every p, which is evaluated to *false* under v, is substituted with $\bot$.

4. Consequence vs. correctness of inference

Suppose we have a definition $\mathbb{D}$ with an associated relation of definitional (=semantic) consequence $\models_{\mathbb{D}}$. Then consequence (with respect to $\mathbb{D}$) and correctness of inference (with respect to $\mathbb{D}$) coincide, if

$$A \models_{\mathbb{D}} B \quad \Leftrightarrow \quad (\models_{\mathbb{D}} A \quad \Rightarrow \quad \models_{\mathbb{D}} B)\ . \tag{10}$$

The direction from left to right of (10), i.e.

$$A \models_{\mathbb{D}} B,\ \models_{\mathbb{D}} A \quad \Rightarrow \quad \models_{\mathbb{D}} B\ , \tag{11}$$

says that consequence ensures correctness of inference. The direction from right to left of (10), i.e.

$$(\models_{\mathbb{D}} A \ \Rightarrow \ \models_{\mathbb{D}} B) \quad \Rightarrow \quad A \models_{\mathbb{D}} B\ , \tag{12}$$

says that correctness of inference establishes consequence. For reasons to be explained later, we call, following Hallnäs [3] a definition $\mathbb{D}$ satisfying (11) *total*, and a definition $\mathbb{D}$ satisfying (12) *complete* (the latter in a sense somewhat different from Hallnäs's). A definition $\mathbb{D}$, which is both total and complete, is called *standard*, as it underlies standard semantics.

By formally distinguishing consequence and correctness or inference, we can formulate conditions for $\mathbb{D}$, under which totality and/or completeness hold. This gives us a possibility to deal with phenomena which are excluded, if consequence and correctness of inference are identified. The main example are non-wellfounded phenomena such as the paradoxes. For example, if we define a by its own negation (with $\bot$ being an undefined object)

$$\mathbb{D}\ \{\ a \ := \ (a \to \bot)$$

we obtain the semantical derivations

$$\begin{array}{cl} a \models a & \\ \hline a,(a\to\bot) \models \bot & \\ \hline a,a \models \bot & (aL) \\ \hline a \models \bot & \\ \hline \models a \to \bot & \\ \hline \models a & (aR) \end{array} \qquad \text{and} \qquad \begin{array}{cl} a \models a & \\ \hline a,(a\to\bot) \models \bot & \\ \hline a,a \models \bot & (aL) \\ \hline a \models \bot & \end{array}$$

which show that both $a \models_{\mathbb{D}} \bot$ and $\models_{\mathbb{D}} a$ hold. Since $\models_{\mathbb{D}} \bot$ does not hold, as there is no definitional clause for $\bot$ in $\mathbb{D}$, this means that $\mathbb{D}$ is not total. It is actually not necessary to require that $\bot$ be undefined, but only that $\bot$ be underivable with respect to $\mathbb{D}$.

A sufficient condition, under which totality holds, is, for example, that defining conditions do not contain (structural) implication. In logic programming, this corresponds to the case of definite programs (whereas the previous example corresponds to a non-definite normal program). Another sufficient condition would be the wellfoundedness of definitional chains, which in logic programming corresponds to stratified normal programs. Other conditions refer to the inference machinery used to generate consequences from definitions. If we disallow contraction in antecedents of consequence statements, we can enforce totality, which corresponds to well-known strategies of avoiding paradoxes proposed by Curry, Fitch and others. Another such condition is to prohibit consequence statements of the form $a \models a$, if a is defined, i.e., if there is a definitional clause for a in $\mathbb{D}$. The rationale behind the latter condition, which was proposed by Kreuger [7], is that if a is defined, consequences involving a should be obtained according to their meaning, i.e., by means of the rules based on the definitional clauses for a (which are definitional closure and definitional reflection with respect to a). This is related to certain three-valued semantics for normal logic programs and related proof-theoretic accounts (see [18] and the references therein).[12]

A definition, which is total, but not complete, is the above definition of the intuitionistic logical constants. It can easily be shown that

$$\models_{\mathbb{LI}} \neg p \supset q \vee r \quad \Rightarrow \quad \models_{\mathbb{LI}} (\neg p \supset q) \vee (\neg p \supset r) \ ,$$

but it can also be shown that

$$\neg p \supset q \vee r \models_{\mathbb{LI}} (\neg p \supset q) \vee (\neg p \supset r)$$

does *not* hold. This example essentially expresses the fact that in intuitionistic propositional logic not all admissible rules are derivable, which was first pointed out by Harrop using this (and other) examples.[13] A nontrivial example of a definition which is both total and complete is the logic $\mathbb{LC}$ of truth and falsity defined in section 3.3.

The reason for calling a definition for which (11) holds *total*, is that it is sensible to call definitions, for which (11) does *not* hold, *partial*. Any definition conveys some

[12]It should be noted that in the latter two cases, in which the inference machinery is changed in such a way that totality is always satisfied, we are not again identifying consequence with correct inference. Though in these cases, consequence coincides with correctness of inference, this is a *result* which can be proved and not something forced by definition as in standard semantics.

[13]An artificial, but much simpler example is the following: Let $\mathbb{D}$ consist of the clauses $a := a$ and $b := b$. This definition is total. Since $\models a$ is not derivable, $(\models a \ \Rightarrow \ \models b)$ is valid. However, $a \models b$ is not derivable.

meaning to the object defined, even if this meaning is very rudimentary and therefore 'partial'. If we are successful, i.e., if consequence based on the given meaning ensures correctness of inference, our meaning assignment is total. However, this success, which might be intended, is not necessarily a syntactic property of the definition[14], but something which must be proved to hold. In this sense it is, as Hallnäs has pointed out, related to the usage of "partial" and "total" in recursive function theory. Whether a partially recursive function is total, is not a syntactic property of its definition — totality, which corresponds to the halting problem, is even undecidable. As with the definition of a partially recursive function, we have an elementary description of what a definition looks like, where the correctness of a definition is not an immediate feature of its being well-formed. The converse of totality, i.e., (12), is called completeness, because it expresses completeness with respect to admissibility, i.e., it expresses the fact that every admissible rule is derivable.[15]

Digression on admissibility and derivability. If $\models_{\mathbb{D}}$ is written as the derivability relation $\vdash$ in a formal system, the principle (10), which is criticized in this paper, is strongly reminiscent of the identification of derivability with admissibility. $A \vdash B$ expresses that the rule $\dfrac{A}{B}$ is derivable, whereas $(\vdash A \;\Rightarrow\; \vdash B)$ expresses that it is admissible. That these notions do not always coincide, is elementary knowledge. In fact, we want to introduce an analogous distinction at the semantical level. There the notion of a rule is played by the notion of a consequence statement $A \models B$, its 'derivability' in a system $\mathbb{D}$ by $A \models_{\mathbb{D}} B$, and its 'admissibility' by $(\models_{\mathbb{D}} A \;\Rightarrow\; \models_{\mathbb{D}} B)$. In standard semantics based on the transmission view of consequence, 'derivability' is defined as 'admissibility' and therefore identified with it, whereas we want to keep these concepts apart. However, the analogy with the syntactical case is only partial: There derivability is the stronger concept which trivially implies admissibility, whereas at the semantical level, 'derivability' does not even imply 'admissibility'.

5. Our model of inference

According to what we have now proposed, consequence is an independent semantical concept which does not necessarily coincide with the correctness of inference and therefore cannot, even in an inferential approach, defined that way. In this sense, it does not necessarily result from inference nor can it guide inference. If the definition considered is not total, consequence would not even imply correctness of inference. Consequence

[14]Sometimes it is, for example in the case of wellfoundedness.

[15]So we associate with "completeness" a slightly different meaning from that given to it by Hallnäs [3], who associated with it the fact that for every object a, either $\models_{\mathbb{D}} a$ or $a \models_{\mathbb{D}} \bot$ holds (with $\bot$ being undefined). This fact is classically related to completeness in our sense. However, our weaker reading of "completeness" seems to us to be more natural, as it is directly related to constructive reasoning.

can be used to guide inferences only if it has been shown beforehand that the definition in question is total.

Now for us consequence is itself an inferential notion. That a consequence statement $A \models_{\mathbb{D}} B$ holds is established by a semantical derivation yielding $A \models B$. There is no need to distinguish any further concept of derivation from this one. An inference step from A to B with respect to a definition $\mathbb{D}$:

$$\frac{A}{B}$$

would have to be written explicitly as a step

$$\frac{\models A}{\models B}$$

in a semantical derivation, where the antecedents of semantical sequents are empty. Its correctness means that if $\models_{\mathbb{D}} A$, then $\models_{\mathbb{D}} B$. What is changed as compared to the standard view is that this correctness cannot always (i.e., if $\mathbb{D}$ is not total and complete) be expressed by the consequence statement with the premiss A as its antecedent and the conclusion B as its consequent: $A \models_{\mathbb{D}} B$.

As for us inference steps are steps in semantical derivations, they actually have the more general form

$$\frac{A_1 \models B_1 \quad \ldots \quad A_n \models B_n}{C \models D} \tag{13}$$

where antecedents are not necessarily empty. This corresponds to the idea that in natural deduction, derivations can depend on assumptions. Here this dependency is expressed by non-empty antecedents, as is the procedure of the sequent calculus. Our model of inference is the sequent-calculus model ([19]). In this more general form, the correctness of an inference step of form (13) (with respect to a definition $\mathbb{D}$) means that, whenever $A_1 \models_{\mathbb{D}} B_1, \ldots,\ A_n \models_{\mathbb{D}} B_n$, then $C \models_{\mathbb{D}} D$.

More precisely, to show that (13) is correct (with respect to $\mathbb{D}$), we have to show that, given grounds for for the premisses, we have to provide (= to construct) grounds for the conclusion. If we denote that g is a ground for $A \models B$ by $g : A \models B$, we can reformulate (13) as

$$\frac{g_1 : A_1 \models B_1 \quad \ldots \quad g_n : A_n \models B_n}{g : C \models D} \tag{14}$$

where it is understood that g is obtained by some constructive operation from $g_1, \ldots, g_n$, i.e., $g = f(g_1, \ldots, g_n)$. We leave it open how to formalize grounds and their handling.

Important here is that a ground g for $A \models B$ codifies a semantical proof of $A \models B$, i.e. (14) expresses that from proofs of the premisses $A_1 \models_{\mathbb{D}} B_1, \ldots, A_n \models_{\mathbb{D}} B_n$ we can construct a proof of the conclusion $C \models_{\mathbb{D}} D$. In this sense the correctness of an inference step expresses the admissibility of this step as an inference rule with respect to the basic semantical inference rules. Semantical steps such as definitional closure and reflection are just a special kind of inference steps, namely those which are immediately justified. "Immediately" means that the grounds for the premisses, which are semantical derivations of them, are extended by an additional step to yield a semantical derivation and therefore ground of the conclusion. For steps which are not primitive steps an admissibility procedure must be given.

For example, take the consequence step

$$\frac{p \models (q \supset r)}{q \models (p \supset r)} \; . \tag{15}$$

If we indicate, quite informally, the grounds for the respective statements by g and g', it reads as:

$$\frac{g \; : \; p \models (q \supset r)}{g' \; : \; q \models (p \supset r)} .$$

Generating g' from g can be done in an obvious way, checking the ways the premiss can be proved in the semantical sequent calculus.

This idea differs from standard proof-theoretic semantics, where inferences are justified by certain transformation procedures, in two respects: Semantical derivations are not just derivations based on introduction inferences (or Right-introduction rules), secondly they are based on sequent-style rather than natural deduction derivations. This way of proceeding makes things simpler in so far as no assumptions need to be considered which may be discharged. As the handling of assumptions is built into sequents, the correctness of an inference of form (13) is a simple admissibility statement, whereas in natural-deduction based proof-theoretic semantics complicated justification procedures must be used.[16]

It should be emphasized that the handling of grounds in the sense described is different from that of terms in the typed λ-calculus. When generating grounds from grounds according to (14) (and elucidated in the example just given), we consider grounds for whole sequents, whereas in the typed λ-calculus terms representing such

[16]It should be mentioned that categorial proof theory is nearer to the semantically understood sequential paradigm than standard proof-theoretic semantics, which is focused on natural deduction. It would thus be worthwhile to investigate the sequential approach to semantics, with its conceptual priority of hypothetical judgements, in categorial terms. The anonymous reviewer reminded me of this fact, pointing in particular to the subobject relationship as the consequence relation in a topos.

grounds are handled within sequents. So the notation $g : A \models B$ we used above, which is understood as $g : (A \models B)$, differs from the λ-calculus notation $x : A \vdash t : B$, where t represents a proof of B from A and the declaration $x : A$ on the left side represents the assumption A. The λ-calculus view and the corresponding Curry-Howard-interpretation actually incorporates the placeholder view of assumptions by always using a variable to represent the ground for an assumption, which by means of substitution can be filled with a term standing for a closed proof of it.

6. Reasoning and the paradox of inference

In the dicussion about logical inference, a so-called "tension" between informativeness and validity has been discussed, notably by Dummett [2] with reference to Cohen and Nagel, but many other authors discuss this issue using different terms. Here by informativeness it is meant that a logical inference should give some new information, otherwise it would be useless, and by validity it is meant that it follows from what is given in the premisses, i.e., without addition of anything from outside. This problem also applies to the case of material inference discussed here, as material inference is inference with respect to a definition $\mathbb{D}$, and should not draw conclusions not contained in $\mathbb{D}$. The solution proposed by Dummett runs roughly as follows: There is some basic sort of inferences, viz. the introduction inferences in natural deduction, which are canonical and which are immediately obvious, as they are (in our terminology) just definitional. However, the second sort of inferences are based on some sort of justification which generates grounds for the conclusion from grounds of the premisses. These are essentially the elimination inferences, and the generation of grounds is done via reduction procedures of the kind elaborated by Prawitz [13].

This solution is not so much different in the case considered here. We also consider certain inferences as basic which constitute the grounds for a sentence or for a consequence statement. These are the semantical inference rules, notably the rules of definitional closure and definitional reflection, which define consequence statements. When performing an inference step, there is a step of generating grounds, which is constituted by certain operations on semantical derivations, where these semantical derivations have now formally the form of (cut-free) sequent-style proofs. So far the pattern is similar: There are canonical inferences constituting grounds, and there are non-canonical inferences obtained by procedures generating grounds from grounds.

The basic difference is that we are now working in a sequent-style framework, where the canonical (definitional) inferences, our 'rules of consequence', comprise both the Right- and Left-inferences. In this sense, assertions and assumptions are on par. The basic entities are hypothetical and not categorical judgements. We change the notion of "canonical" (or "definitional", as we call it), but not the idea that there must be justification procedures for those inferences that do not follow the definitional pattern.

As we do have the distinction between Left- and Right-inferences, one may now even draw a twofold distinction: Between Right- and Left-inferences on the one hand and between definitional (i.e., Right- plus Left-) and non-definitional inferences.[17] This twofold distinction is justified, as already the Left-inferences are in some way indirect because they involve a step of reflection upon the definition as a whole and not just follow single definitional clauses as do the Right-inferences. An inference of the form (13) may thus be either a direct definitional inference (based on definitional closure) or an indirect definitional inference (based on definitional reflection) or a non-definitional inference (based on a procedure generating grounds from grounds). Whereas direct and indirect definitional inferences are meaning giving and thus do not generate new information, a non-definitional inference such as (15) is based on the generation of grounds and is therefore informative.[18]

The approach sketched here based on the primacy of hypothetical judgements can be carried over to other reasoning formats as well. In natural deduction style it would lead to bidirectional natural deduction [21]. Another framework would be the dialogical or game-theoretical approach to semantics, in which the duality of assumptions and assertions is reflected by the roles of the two players of a game, which means that the primacy of the hypothetical is built into the framework from the very beginning. A further advantage of all approaches based on hypothetical reasoning is the lack of functionals to interpret iterated implications, which are necessary in standard constructive semantics, but which, in particular in combination with non-wellfounded definitions, pose severe problems when they are interpreted epistemologically.

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[17] One might perhaps even introduce a threefold distinction, where one also separates the structural principles from the semantical principles in the narrower sense (definitional closure and definitional reflection).

[18] We are claiming that it is the generation of justifications or grounds which makes an inference informative. This need not always be a generation of grounds for every single step 'from scratch'. In the case where we have totality, it may also establish some consequence and then use this consequence in an inference using cut. The generation of grounds would then be hidden in the proof of this consequence together with the cut elimination proof for the whole system.

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