

Frege and the Resolution Calculus

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We reconstruct Frege's treatment of certain deducibility problems posed by Boole. It turns out that in his formalization and solution of Boole's problems Frege anticipates the idea of propositional resolution.

1. Boole's example

In his posthumously published paper 'Booles rechnende Logik und die *Begriffsschrift*' ('Boole's computational logic and the *Begriffsschrift*'), written in 1880 or 1881, which was motivated by Schröder's (1880) review of the *Begriffsschrift*,¹ Frege compares his novel approach to logic with Boole's and with Schröder's algebra of logic. He basically defends his functional view of concepts and his account of quantification, and demonstrates the expressive power of his system by many examples, including nested quantifiers and such complicated operations as the substitution of formulas for function variables in quantified formulas. Although Frege is well aware of the fact that the handling of functions and of quantification is the main achievement of his *Begriffsschrift* (51/45), he demonstrates by means of an example from Boole's *Laws of Thought* (1854) that he can deal as well with problems of propositional logic which in the algebra of logic are solved using equational reasoning. As he says:

It would not be surprising and I could happily concede the point, if Boolean logic were better suited than my *Begriffsschrift* to solve the kind of problems it was specifically designed for, or for which it was specifically invented. But maybe not even this is the case (44/39).

Boole's example had been independently considered by Schröder (1877), by Wundt (1880), who essentially followed Schröder's presentation, and also by Lotze (1880). Frege refers to Boole, Schröder and Wundt.² Boole's own formulation was as follows:

Let the observation of a class of natural productions be supposed to have led to the following general results.

1st, That in whichever of these productions the properties *A* and *C* are missing, the property *E* is found, together with one of the properties *B* and *D*, but not with both.

- 1 For the history of Frege's paper see *Nachgelassene Schriften* (1969), p. 9, footnote 1; *Posthumous Writings* (1979), p. 9, footnote 1. In the following, page numbers of the form *n/m* refer to this paper by Frege, where *n* is the page number in *Nachgelassene Schriften* and *m* the corresponding page number in the translation *Posthumous Writings*.
- 2 Gottfried Gabriel, to whom I owe the reference to Lotze, suggests that Frege perhaps missed the passage in Lotze, which is part of an appendix added to the third chapter in the second edition of his *Logik* (pp. 265–267) – see Gabriel 1989.

2nd, That wherever the properties A and D are found while E is missing, the properties B and C will either both be found, or both be missing.

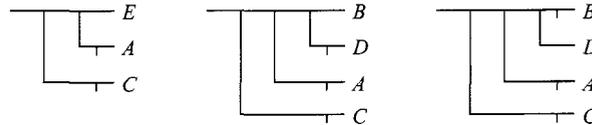
3rd, That wherever the property A is found in conjunction with either B or E , or both of them, there either the property C or the property D will be found, but not both of them. And conversely, wherever the property C or D is found singly, there the property A will be found in conjunction with either B or E , or both of them.

Let it then be required to ascertain, first, what in any particular instance may be concluded from the ascertained presence of the property A , with reference to the properties B , C , and D ; also whether any relations exist independently among the properties B , C , and D . Secondly, what may be concluded in like manner respecting the property B , and the properties A , C , and D (Boole 1854, p. 146).

This paper attempts to show that Frege's treatment of Boole's example anticipates for the propositional case certain ideas which since Robinson (1963, 1965) are commonly grouped under the heading 'resolution'. For this purpose we take a fresh look, from a more modern point of view, at Frege's line of argument.

2. Begriffsschrift implications as clauses

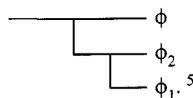
Frege translates the three conditions of Boole's example into a collection of implicational formulas of propositional logic. For example, the first condition is translated into the following three implications:³



The problems posed by the example then consist in determining which conclusions of a certain form can be logically deduced from these conditions. For example, the first problem⁴ ('what in any particular instance may be concluded from the ascertained presence of the property A , with reference to the properties B , C , and D ') is to determine which implication holds with A occurring in the antecedent and with at most B , C and D occurring in the implication in addition to A . Frege considers only implications of the form



without nesting of implications to the left, as in



3 I use Latin letters throughout, not distinguishing (as Frege did) between the description of the problem (Latin letters) and its logical representation (Greek letters).
 4 Each of the two tasks in Boole's example consists of two subtasks, so we have four problems.

When dealing with a formula of the form (*) Frege leaves it ambiguous, whether ϕ_1, \dots, ϕ_n are to be considered conjunctively, such that (*) is to be read as $(\phi_1 \wedge \dots \wedge \phi_n) \rightarrow \phi$, or whether (*) represents an iterated implication of the form $\phi_1 \rightarrow (\dots (\phi_n \rightarrow \phi) \dots)$. This ambiguity, which is justified since both readings are logically equivalent, is a great but rarely acknowledged⁶ advantage of Frege's notation. It allows us to write (*) in the form of a sequent

$$(**) \phi_1, \dots, \phi_n \Rightarrow \phi$$

in Gentzen's sense with a single formula as the succedent.

Since Frege permits the contraction ('Verschmelzung' in the terminology of his *Grundgesetze der Arithmetik*) of two occurrences ϕ_i and ϕ_j of the same formula (i.e., $\phi_i = \phi_j$) as well as arbitrary permutation ('Vertauschung' in *Grundgesetze*) of ϕ_1, \dots, ϕ_n , we may consider the antecedent of (**) as a set of formulas. Furthermore, since there is no nesting of implication to the left in the present context, $\phi_1, \dots, \phi_n, \phi$ are *literals*, i.e. either atomic formulas (propositional variables, *atoms*) or negations thereof. Finally, using his contraposition laws ('Wendung' in *Grundgesetze*), Frege identifies

$$\phi_1, \dots, \phi_n \Rightarrow \phi$$

with any

$$\bar{\phi}, \phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_n \Rightarrow \bar{\phi}_i.$$

Here $\bar{\phi}$ is $\neg\phi$ if ϕ is an atom, and ψ if ϕ is of the form $\neg\psi$. Thus Frege permits exchanging a member ϕ_i of the antecedent ('Bedingung'/'condition') with the succedent ('Folge'/'consequence') ϕ by replacing them at the same time with their opposites $\bar{\phi}_i$ and $\bar{\phi}$.⁷

This identification entitles us to use multiple-succedent sequents of *atoms* (i.e. sequents with a finite set of atoms on the right and on the left side of \Rightarrow) as invariant representatives of single-succedent sequents modulo contraposition. This means that the sequent

$$\alpha_1, \dots, \alpha_n \Rightarrow \beta_1, \dots, \beta_m$$

with *atoms* α_i, β_j stands for any sequent of the form

$$\alpha_1, \dots, \alpha_n, \neg\beta_1, \dots, \neg\beta_{i-1}, \neg\beta_{i+1}, \dots, \neg\beta_m \Rightarrow \beta_i$$

or

$$\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n, \neg\beta_1, \dots, \neg\beta_m \Rightarrow \neg\alpha_i.$$

In this way the sequent notation can dispense with the negation sign. In the following, α and β (with and without indices) stand for atoms exclusively.

The sequent

$$\alpha_1, \dots, \alpha_n \Rightarrow \beta_1, \dots, \beta_m$$

can also be read as

$$(\alpha_1 \wedge \dots \wedge \alpha_n) \rightarrow (\beta_1 \vee \dots \vee \beta_m)$$

or as

$$\neg\alpha_1 \vee \dots \vee \neg\alpha_n \vee \beta_1 \vee \dots \vee \beta_m.$$

5 I use lower case Greek letters as syntactical variables for formulas.

6 An exception is Thiel 1995.

7 We prefer Gentzen's technical terms 'antecedent' and 'succedent' to Frege's terminology, particularly since in the following we shall consider sequents as consequences of other sequents.

The latter formula is normally called a *clause* in the theory of resolution, sometimes being formulated as a set which is understood disjunctively:⁸

$$\{\neg\alpha_1, \dots, \neg\alpha_n, \beta_1, \dots, \beta_m\}.$$

We follow this terminology, calling a sequent of atoms

$$\alpha_1, \dots, \alpha_n \Rightarrow \beta_1, \dots, \beta_m$$

a *clause* (as is common in proof-theoretic approaches to resolution⁹). The antecedent or the succedent of a clause may be empty. As a limiting case the empty clause \Rightarrow represents a contradiction.

For example, clausal translations of the three *Begriffsschrift* implications mentioned at the beginning of this section are

$$\begin{aligned} &\Rightarrow A, C, E \\ &\Rightarrow A, B, C, D \\ B, D &\Rightarrow A, C. \end{aligned}$$

Using clausal terminology, Boole's first problem asks which clauses of the form $A, \Gamma \Rightarrow \Delta$ ¹⁰ are derivable from clausal translations of the assumptions, where Γ and Δ contain *at most* B, C and D , and similarly with the other problems.

In our clausal framework, which hides rules of contraposition, Frege's way of proceeding from implications to implications can be viewed as deriving clauses from clauses in the calculus of (propositional) resolution, as will be shown in the following.

3. Propositional resolution

Frege argues that in order to obtain a solution to any of Boole's problems, we have to *eliminate* ('wegschaffen') certain atoms from the clauses assumed. For the first problem, the clauses $A, \Gamma \Rightarrow \Delta$ to be deduced must not contain E . Therefore, if we want to use assumptions in which E occurs (most of Boole's assumptions actually contain E), we have to eliminate E from them. According to Frege, 'this can be done if E is a consequence in one judgement [...] and a condition in another [...]' (47/42) or 'where its affirmation is a condition in one and its denial in the other' (48/42), the second principle being a variant of the first one in view of contraposition. Using clause notation, the underlying law is the rule of propositional resolution:

$$(Res) \frac{\Gamma_1 \Rightarrow \Delta_1, \alpha \quad \alpha, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \text{ provided } \alpha \notin \Delta_1 \text{ and } \alpha \notin \Gamma_2,$$

which (unlike the full resolution rule with unification) is an atomic case of Gentzen's cut rule. Here α is called the *resolution atom* of the application of (Res) considered. Its conclusion is called *the resolvent of the premises with respect to α* . If the two premises belong to a set, K , of clauses, the conclusion is also called *a resolvent of K with respect to α* . A resolution step with resolution atom α is also called an α -*resolution*. In the following it is always tacitly understood that in applications of (Res) the proviso stated in the formulation of the rule is satisfied.

8 See e.g. Gallier 1986.

9 See e.g. Robinson 1979.

10 We use capital Greek letters for finite sets of atoms. As usual, Γ, A and Γ, Δ , etc. stand for the sets $\Gamma \cup \{A\}$ and $\Gamma \cup \Delta$, etc., respectively.

For example, (*Res*) allows us to eliminate E from

$$A, C, D \Rightarrow B, E \quad [\neg E, D, A, \neg B \Rightarrow \neg C]$$

and

$$A, C, D, E \Rightarrow [E, A, D \Rightarrow \neg C]$$

(Frege's single-succedent formulation in brackets),¹¹ yielding

$$A, C, D \Rightarrow B$$

as a resolvent, which is already one solution to Boole's first problem.

It is crucial for our interpretation that Frege considers his principle of the elimination of atoms—the resolution rule—not just as one possible inference rule justifiable in the logical system of the *Begriffsschrift*, but as a rule that is sufficient to generate *all* solutions to Boole's problems (and, of course, to related problems as well). Frege explicitly claims that he is able to obtain *complete answers* (48/43). In this way Frege does not anticipate the *resolution method* as a refutation procedure, according to which the validity of a formula ϕ is established by deriving the empty clause \Rightarrow from a set of clauses $\{\Gamma_1 \Rightarrow \Delta_1, \dots, \Gamma_n \Rightarrow \Delta_n\}$ that represents a conjunctive normal form of $\neg\phi$. However, Frege does anticipate the *resolution calculus*, which is based on the idea of deriving clauses from other clauses by means of the rule of resolution as *the basic rule of inference*. This also means that, in a certain sense, we find in Frege already the idea of a rule-based logic normally attributed to Gentzen (albeit, of course, only in a very restricted form).

The resolution calculus Frege implicitly uses can be described as a system \mathcal{R} for the derivation of clauses from clauses by means of tautology axioms of the form

$$(Taut) \alpha, \Gamma \Rightarrow \Delta, \alpha,$$

the resolution rule (*Res*), and the rule of thinning

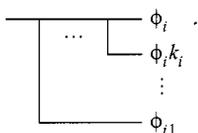
$$(Thin) \frac{\Gamma_1 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}.$$

We have to add (*Thin*) and (*Taut*) to (*Res*), since we are not interested merely in derivations of the empty clause.¹²

According to our interpretation, Frege implicitly assumes something like the following lemma concerning the relationship between the system of his *Begriffsschrift* and the calculus used in the discussion of Boole's example. Let $\vdash_{\mathcal{B}}$ denote derivability in the formal system of the *Begriffsschrift*. Let $\vdash_{\mathcal{R}}$ denote derivability in the resolution calculus based on (*Taut*), (*Thin*) and (*Res*). Let $\theta_1, \dots, \theta_n, \theta$ be *Begriffsschrift* implications, each of the form

11 These are conditions (5) and (9); see section 4.

12 The precise formulation of (*Taut*) and (*Thin*) is a matter of technical convenience. For example, due to the presence of thinning, we could have chosen tautology axioms of the form $\alpha \Rightarrow \alpha$. Frege is aware of these additional principles. For thinning this is shown by the fact that he considers, e.g., the clause $A, C, D \Rightarrow$ to be a solution to the first problem, although it does not contain B (i.e., from which a solution containing B is obtained by thinning) (see 48/43). For tautologies this can be concluded from his taking into account only implications which 'give information' ('geben Auskunft') about certain contents (47/42), where those implications, which do not give any information, have the form of our tautology axioms.



for literals $\phi_{i1}, \dots, \phi_{ik_i}, \phi_i$ ($1 \leq i \leq n$). Let $\theta_1^*, \dots, \theta_n^*, \theta^*$ be clausal translations thereof. Conversely, given a clause κ , let κ^+ denote any *Begriffsschrift* implication of the form just mentioned, which represents κ , i.e., of which κ is a clausal translation. Then the following holds:

Lemma 1

$$\begin{aligned} \theta_1, \dots, \theta_n \vdash_{\mathcal{R}} \theta & \text{ iff } \theta_1^*, \dots, \theta_n^* \vdash_{\mathcal{R}} \theta^* \\ \kappa_1, \dots, \kappa_n \vdash_{\mathcal{R}} \kappa & \text{ iff } \kappa_1^+, \dots, \kappa_n^+ \vdash_{\mathcal{R}} \kappa^+. \end{aligned}$$

The proof is easy.¹³

Using certain proof-theoretic results about \mathcal{R} , we shall justify further features of Frege's solution to Boole's problems and finally rigorously establish that Frege's solution is in fact complete. We use throughout our clausal translation of Frege's terminology.

Frege's procedure to obtain all consequences of a certain form of a set of clauses is the following (47–8/42–3). First he checks whether any of the assumptions is already a solution to the problem under consideration. For instance, in solving Boole's first problem, Frege looks whether a clause of the form $A, \Gamma \Rightarrow \Delta$, with E not occurring in Γ or Δ , is already among the assumptions. In the next step he chooses an atom, α , which is not to occur in the solution to the problem and generates all resolvents with respect to α of the set of clauses given, thus eliminating α . For Boole's first problem this means that Frege has to perform all possible resolutions with E as the resolution atom. If necessary, he eliminates some further atom in the next step, based on the clauses he has obtained in the previous step. In the solution to Boole's first problem, he computes all resolvents with respect to B of the clauses without E obtained so far, since a sequent $A, \Gamma \Rightarrow \Delta$, with Γ and Δ only containing C and D , would also be a solution to the problem, and so on.

In proceeding in this way, Frege considers only resolution steps in which the resolvent is not tautological, arguing that tautologies do not convey any substantial information and that we are interested only in non-tautological solutions to our problems.

To justify this procedure and thus Frege's claim that in his paper he has obtained *all* solutions to the problems posed, we have to show that we can always proceed by eliminating atoms *step by step* and that non-tautological resolvents are useless in the derivation of a non-tautological clause.

Define a subclause κ' of a clause κ to be a clause, from which κ can be obtained by (*Thin*), with κ being a subclause of itself as a limiting case. Let σ be a sequence of atoms, let $\langle \rangle$ be the empty sequence, and let $\sigma * \alpha$ be $\langle \alpha_1, \dots, \alpha_n, \alpha \rangle$, if σ is $\langle \alpha_1, \dots, \alpha_n \rangle$ (where $\langle \rangle * \alpha$ is $\langle \alpha \rangle$). Let K be any finite set of clauses. Let K_α be the set of those clauses in K which contain α . Let $Rs^\alpha(K)$ be the set of all non-tautological resolvents

¹³ If one chooses the propositional fragment of the *Grundgesetze* as the basis for $\vdash_{\mathcal{R}}$, the proof is straightforward, since this system has axioms corresponding to (*Taut*) and inference rules corresponding to (*Res*) as the only principles of deduction apart from the rules hidden in our usage of multiple-succedent sequents.

of K with respect to α . The restriction of $Res^\alpha(K)$ to non-tautological clauses guarantees in particular that α does not occur in $Res^\alpha(K)$, provided K does not contain tautological clauses.¹⁴ For a sequence of atoms σ let $Res^\sigma(K)$ be defined as follows:

$$Res^\diamond(K) = \text{the set of non-tautological clauses in } K$$

$$Res^{\sigma^*\alpha}(K) = (Res^\sigma(K) \setminus (Res^\sigma(K))_\alpha) \cup Res^\alpha(Res^\sigma(K)).$$

This means that $Res^{\sigma^*\alpha}(K)$ contains, first, all clauses in $Res^\sigma(K)$ which do not contain α ; and, second, all non-tautological resolvents of clauses in $Res^\sigma(K)$ with respect to α .¹⁵

Then we have the following result:

Lemma 2

- (i) $Res^\sigma(K) = Res^{\sigma'}(K)$, if σ' is a permutation of σ .
- (ii) $K \vdash_{\mathcal{R}} \kappa$ for non-tautological κ iff there is a subclause κ' of κ such that $\kappa' \in Res^\sigma(K)$, where σ is the set of all atoms which are in K but not in κ' .

The proof is sketched in the appendix. Part (i) says that the order of eliminations of atoms is irrelevant, so that we may consider σ to be a set. Therefore, in the following we use a terminology like $Res^\emptyset(K)$ or $Res^{\sigma \cup \tau}(K)$ for *sets* of atoms σ or τ . Part (ii) says that, if τ is the set of atoms in K , then by generating

$$\bigcup \{Res^\sigma(K) \mid \sigma \subseteq \tau\}$$

we receive a (finite) set of clauses, from which by thinning we obtain all non-tautological clauses derivable from K . This finite set may be viewed as *representing* the (infinite) set of non-tautological consequences of K .¹⁶ For example, if in K the atoms α , β and γ occur, this set is

$$Res^\emptyset(K) \cup Res^{\{\alpha\}}(K) \cup Res^{\{\beta\}}(K) \cup Res^{\{\gamma\}}(K)$$

$$\cup Res^{\{\alpha, \beta\}}(K) \cup Res^{\{\alpha, \gamma\}}(K) \cup Res^{\{\beta, \gamma\}}(K) \cup Res^{\{\alpha, \beta, \gamma\}}(K).$$

Since κ' is a subclause of κ , it furthermore follows from Lemma 2 that, if τ' is the set of atoms in K but not in κ ,

$$\bigcup \{Res^\sigma(K) \mid \tau' \subseteq \sigma \subseteq \tau\}$$

is a set from which, by thinning, we can derive all non-tautological clauses not containing atoms from τ' . Therefore, this set may be viewed as representing all non-tautological consequences of K without atoms from τ' . For example, if in K the atoms α , β and γ occur, then the set

$$Res^{\{\alpha\}}(K) \cup Res^{\{\alpha, \beta\}}(K) \cup Res^{\{\alpha, \gamma\}}(K) \cup Res^{\{\alpha, \beta, \gamma\}}(K)$$

represents the set of all non-tautological consequences of K which do not contain α .

Furthermore, if we are looking for clauses of the form $\alpha, \Gamma \Rightarrow \Delta$ (with α not in Γ or Δ), which are derivable from K such that $\Gamma \Rightarrow \Delta$ alone is not derivable from K , then

14 That an atom occurs in K means that it occurs in some clause in K .

15 We might further restrict $Res^{\sigma^*\alpha}(K)$ by removing a clause κ if a subclause κ' of κ already occurs in this set (see e.g. Robinson 1979, p. 201). However, we refrain from this here. -The operation Res^σ is a generalization of the usual closure operation for resolution steps. The usual one does not specify the resolution atom used in each step (see Goltz and Herre 1990, pp. 43–6).

16 Therefore, Lemma 2 shows that what is actually used to *compute* consequences of a set of clauses K , is resolution *only*. Tautology axioms and thinning are just closure conditions which serve the purpose to generate *all* clauses including non-informative ones, i.e. those without content at all (tautologies) as well as those which are just weakenings of informative clauses.

from (ii) it follows that we need not compute any $\text{Res}^\sigma(K)$ with α in σ . Since in such computations an α occurring on the right side of \Rightarrow always remains on that side and never disappears, we can disregard any clause in K and any resolution step in which α occurs on the right side of \Rightarrow . The analogue holds, of course, for clauses of the form $\Gamma \Rightarrow \Delta, \alpha$.

4. Computation of the solutions

Now we can reconstruct Frege's solution to Boole's problems in detail. If we translate the *Begriffsschrift* implications, by means of which Frege formalizes Boole's assumptions, into clauses, we obtain the following:

- (1) $\Rightarrow A, C, E$
- (2) $\Rightarrow A, B, C, D$
- (3) $B, D \Rightarrow A, C$
- (4) $A, B, D \Rightarrow C, E$
- (5) $A, C, D \Rightarrow B, E$
- (6) $A, B \Rightarrow C, D$
- (7) $A, B, D, C \Rightarrow$
- (8) $A, E \Rightarrow C, D$
- (9) $A, C, D, E \Rightarrow$
- (10) $C \Rightarrow A, D$
- (11) $D \Rightarrow A, C$
- (12') $C \Rightarrow B, D, E$
- (12'') $D \Rightarrow B, C, E$

With the exception of (12') and (12'') the numbering is due to Frege.¹⁷ Clauses (1)–(3) formalize Boole's first assumption; clauses (4) and (5) formalize Boole's second assumption; and clauses (6)–(12''), Boole's third assumption. In the following, K denotes the set of assumptions (1)–(12'').

Boole's problems can be stated as follows:

1. Determine the set of all non-tautological consequences of K of the form $A, \Gamma \Rightarrow \Delta$, such that Γ and Δ contain at most the atoms B, C and D . Here we are interested in clauses in which A 'relevantly' occurs; i.e. for which $\Gamma \Rightarrow \Delta$ is not a consequence of K .
2. Determine the set of all non-tautological consequences of K of the form $\Gamma \Rightarrow \Delta$, such that Γ and Δ contain at most the atoms B, C and D .
3. Determine the set of all non-tautological consequences of K of the form $B, \Gamma \Rightarrow \Delta$, such that Γ and Δ contain at most the atoms A, C and D . Here we are interested in clauses in which B 'relevantly' occurs; i.e. for which $\Gamma \Rightarrow \Delta$ is not a consequence of K .
4. Determine the set of all non-tautological consequences of K of the form $\Gamma \Rightarrow \Delta$, such that Γ and Δ contain at most the atoms A, C and D .

Our numbering of clauses throughout follows Frege's numbering. If m and n are numbers of clauses, by $m \times n$ we denote the resolvent of m and n based on a resolution

17 Instead of (12') and (12''), which are correct, Frege uses the incorrect clause (12) $D \Rightarrow B, E$, as has been pointed out by the editors and by the translators (see footnotes on p. 47/41). However, exchanging (12') and (12'') with (12) causes no difference to the results obtained below, as can easily be checked.

step with m as the left premise and n as the right premise, provided it is defined. From the context it will always be clear which is the resolution atom. Since K is fixed, we suppress the reference to K and write Res^σ rather than $Res^\sigma(K)$.

Solution to problem 1. We have to compute the union of the following sets:

- $Res^{(E)}$
- $Res^{(E, B)}$
- $Res^{(E, C)}$
- $Res^{(E, D)}$
- $Res^{(E, B, C)}$
- $Res^{(E, B, D)}$
- $Res^{(E, C, D)}$
- $Res^{(E, B, C, D)}$

Since K contains no tautological clauses, we have $Res^\diamond = K$. Therefore:

$$Res^{(E)} = (K \setminus K_E) \cup Res^E(K).$$

Since we can disregard clauses in which A occurs on the right side of \Rightarrow , we have:

$$K \setminus K_E = \{(6), (7)\},$$

which gives two solutions already. Resolvents of K with respect to E are

$$(1) \times (8), (4) \times (8), (5) \times (8), (12') \times (8), (12'') \times (8), \\ (1) \times (9), (4) \times (9), (5) \times (9), (12') \times (9), (12'') \times (9)$$

all of which, however, are tautologies, except $(5) \times (9)$. The latter is the clause

$$(16) \quad A, C, D \Rightarrow B,$$

which is the third solution.¹⁸ Therefore:

$$Res^{(E)} = \{(6), (7), (16)\}.$$

Since each clause in $\{(6), (7), (16)\}$ contains B , the set $Res^{(E, B)}$ is obtained by computing $(16) \times (6)$ and $(16) \times (7)$, which are the only resolvents of $Res^{(E)}$ with respect to B . Here $(16) \times (6)$ is a tautology, whereas $(16) \times (7)$ is:

$$(17) \quad A, C, D \Rightarrow.$$

Thus:

$$Res^{(E, B)} = \{(17)\}.$$

The procedure so far is completely parallel to Frege's argumentation (47–8/42–3). In addition to $Res^{(E)}$ and $Res^{(E, B)}$ Frege does not compute any further sets of resolvents. However, he is aware of possible further cases, as is shown by his writing of the elimination of ' B , -say'. It can be seen that all further sets are empty: since each clause in $\{(6), (7), (16)\}$ contains C , we obtain $Res^{(E, C)}$ by computing the resolvents $(6) \times (7)$ and $(6) \times (16)$ with respect to C , which are both tautological. Thus $Res^{(E, C)} = \emptyset$. Similarly, $Res^{(E, D)} = \emptyset$, since all resolvents to be considered are tautological. Since $\{E, C\} \subseteq \sigma$ or $\{E, D\} \subseteq \sigma$ for all remaining Res^σ , each such Res^σ is empty.

18 If we used (12) instead of (12') and (12''), as Frege erroneously does, $(12) \times (9)$ would be a non-tautology, but would be identical to (16). So no different result would arise.

Therefore the complete solution to problem 1 is given by the set:

$$\{(6), (7), (16), (17)\}$$

which is in full accordance with Frege.

Solution to problem 2. We have to compute the union of the following sets:

$$\begin{aligned} & Res^{(E, A)} \\ & Res^{(E, A, B)} \\ & Res^{(E, A, C)} \\ & Res^{(E, A, D)} \\ & Res^{(E, A, B, C)} \\ & Res^{(E, A, B, D)} \\ & Res^{(E, A, C, D)} \\ & Res^{(E, A, B, C, D)} \end{aligned}$$

As before,

$$Res^{(E)} = (K \setminus K_E) \cup Rs^E(K).$$

Since we cannot use the positional argument with respect to A as we did before, we have:

$$K \setminus K_E = \{(2), (3), (6), (7), (10), (11)\}$$

which, together with

$$Rs^E(K) = \{(16)\},$$

yields:

$$Res^{(E)} = \{(2), (3), (6), (7), (10), (11), (16)\}.$$

All clauses in $Res^{(E)}$ contain A . Therefore:

$$Res^{(E, A)} = Rs^A(Res^{(E)}).$$

All resolvents of $Res^{(E)}$ with respect to A turn out to be tautological. Thus $Res^{(E, A)}$ and all other sets to be computed are empty. So the solution to the problem is the empty set. Frege argues in exactly the same way, listing in a table all possible resolutions with resolution atom A (49/43–4).¹⁹

Solution to problem 3. We have to compute the union of the following sets:

$$\begin{aligned} & Res^{(E)} \\ & Res^{(E, A)} \\ & Res^{(E, C)} \\ & Res^{(E, D)} \\ & Res^{(E, A, C)} \\ & Res^{(E, A, D)} \\ & Res^{(E, C, D)} \\ & Res^{(E, A, C, D)} \end{aligned}$$

As before,

$$Res^{(E)} = (K \setminus K_E) \cup Rs^E(K).$$

¹⁹ In this table, the numbers '(17)' and '(18)' have to be replaced with '(18)' and '(19)', respectively (this mistake is found both in the German and English editions). Formula (17) is no candidate for a resolution step at all in the present context, whereas (18) and (19) are *Beschriftsschrift* implications corresponding to (2) and (3), respectively.

Since in (2) B occurs on the right side of \Rightarrow , we may disregard it. Therefore

$$(K \setminus K_E) = \{(3), (6), (7), (10), (11)\}.$$

Since in (16), which is the only non-tautological resolvent of K with respect to E (see above), B occurs on the right side of \Rightarrow , $Rs^E(K)$ is empty. Therefore:

$$Res^{(E)} = \{(3), (6), (7), (10), (11)\}.$$

It can easily be seen that $Res^{(E,A)}$, $Res^{(E,C)}$ and $Res^{(E,D)}$ are empty, since all resolvents to be considered are tautological. Therefore all remaining Res^σ are empty as well. Since (10) and (11) are already solutions to problem 4, we omit them and obtain $\{(3), (6), (7)\}$ as the complete solution to problem 3. This is in accordance with Frege, who (on 49/44) considers only $(K \setminus K_E)$ (disregarding (10) and (11)), without computing any resolvent. Perhaps he considers it obvious that all resolvents are tautological.

Solution to problem 4. We have to compute the union of the following sets:

$$\begin{aligned} &Res^{(E,B)} \\ &Res^{(E,B,A)} \\ &Res^{(E,B,C)} \\ &Res^{(E,B,D)} \\ &Res^{(E,B,A,C)} \\ &Res^{(E,B,A,D)} \\ &Res^{(E,B,C,D)} \\ &Res^{(E,B,A,C,D)} \end{aligned}$$

Since we cannot use arguments concerning the position of B in a clause, we now obtain:

$$Res^{(E)} = \{(2), (3), (6), (7), (10), (11), (16)\}.$$

Then:

$$\begin{aligned} Res^{(E,B)} &= (Res^{(E)} \setminus (Res^{(E)})_B) \cup Rs^B(Res^{(E)}) \\ &= \{(10), (11)\} \cup Rs^B(Res^{(E)}) \end{aligned}$$

The resolvents of $Res^{(E)}$ with respect to B are the following:

$$(2) \times (3), (2) \times (6), (2) \times (7), (16) \times (3), (16) \times (6), (16) \times (7).$$

Only $(16) \times (7)$, which is (17) [i.e. $A, C, D \Rightarrow$], is non-tautological. Thus

$$Res^{(E,B)} = \{(10), (11), (17)\}.$$

This is computed by Frege in exactly the same way (49/44). That the elimination of further atoms (which is not explicitly considered by Frege) does not add anything to the solution can easily be seen as follows: since each clause in $Res^{(E,B)}$ contains A , C and D , the sets $Res^{(E,B,A)}$, $Res^{(E,B,C)}$ and $Res^{(E,B,D)}$ consist of the resolvents of $Res^{(E,B)}$ with respect to A , C and D , respectively. All these resolvents are tautological. Therefore all Res^σ beyond $Res^{(E,B)}$ are empty, which means that $\{(10), (11), (17)\}$ is the complete solution.

The last paragraph of his paper clearly demonstrates that Frege himself is not aware either of the conceptual significance of the problem he is dealing with or of the computational significance of his solution:

It [the *Begriffsschrift*] can be used to solve the sort of problems Boole tackles, and even do so with fewer algorithmic prerequisites. This is the point to which I attach least importance, since such problems will seldom, if ever, occur in science (52/46).

In the meantime Gentzen has shown that the deviation of implicational ('sequent') structures, to which Frege had reduced Boole's problems, lies at the heart of logic, and Robinson has shown that resolution is an extremely powerful rule, the only one needed for such derivations in the clausal case. From this perspective Frege's insights turn out to be much deeper than he himself could have estimated.

Appendix: Proof of Lemma 2

Suppose a derivation \mathcal{D} of κ from K in \mathcal{R} is given, where κ is non-tautological. We first show that all tautological clauses can be eliminated from \mathcal{D} and that all applications of thinning can be moved down to the last step. Then we show that resolutions can be permuted.

Removal of (Taut) and restriction of (Thin) to the last step

An occurrence of a clause in \mathcal{D} is called a *critical clause*, if it is either tautological (not necessarily an axiom!) or the conclusion of an application of (Thin), and at the same time a premise of an application of (Res) or of (Thin). In order to eliminate critical clauses, we perform certain reductions. For example, we replace

$$\frac{\frac{\mathcal{D}_1}{\Gamma_1 \Rightarrow \Delta_1, \alpha} \quad \frac{\mathcal{D}_2}{\alpha, \Gamma_2 \Rightarrow \Delta_2, \alpha}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \alpha} (Res)$$

(here the tautological right premise of (Res) is a critical clause) with

$$\frac{\frac{\mathcal{D}_1}{\Gamma_1 \Rightarrow \Delta_1, \alpha}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \alpha} (Thin).$$

Similarly, we replace

$$\frac{\frac{\frac{\mathcal{D}_1}{\Gamma_1 \Rightarrow \Delta_1, \alpha}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \alpha} \quad \frac{\mathcal{D}_2}{\alpha, \Gamma_3 \Rightarrow \Delta_3}}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3} (Res)$$

(here the left premise of (Res) is a conclusion of (Thin) and therefore a critical clause) with

$$\overline{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3} (Taut)$$

if $\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3$ is tautological, and with

$$\frac{\frac{\frac{\mathcal{D}_1}{\Gamma_1 \Rightarrow \Delta_1, \alpha} \quad \frac{\mathcal{D}_2}{\alpha, \Gamma_3 \Rightarrow \Delta_3}}{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3} (Res)}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3} (Thin)$$

otherwise.²⁰ Other cases of critical clauses are treated analogously. If a new critical clause happens to be generated by these reductions, then its *height* is always lower than

²⁰ As usual, the notation $\frac{\mathcal{D}}{\kappa}$ indicates that \mathcal{D} ends with κ .

that of the critical clause eliminated, where the height of an occurrence of a clause is the number of inference steps between this occurrence and the end clause κ . Therefore, by choosing an appropriate induction measure, we can show that a derivation \mathcal{D}° of κ from K can be obtained which contains no critical clause. If κ is non-tautological, this means that \mathcal{D}° has the form

$$\frac{\mathcal{D}''}{\kappa} (\text{Thin}),$$

where the last step may be lacking, in which case κ'' is κ and \mathcal{D}'' is \mathcal{D}° . The derivation \mathcal{D}'' has the following properties:

- (1) Only non-tautological members of K occur as assumptions.
- (2) Resolution (*Res*) with non-tautological conclusion is the only rule applied (in particular, tautology axioms or thinning are not used at all).
- (3) If an α -resolution is used, then α occurs in K .

Permutation of resolution steps

Suppose a sequence $\langle \alpha_1, \dots, \alpha_n \rangle$ of atoms which occur in K is given. Suppose $1 \leq i, j \leq n$ where $i \neq j$. We can change the order of resolutions in \mathcal{D}'' as in the following example. Replace

$$\frac{\frac{\frac{\mathcal{D}_1}{\Gamma_1 \Rightarrow \Delta_1, \alpha_i, \alpha_j} \quad \frac{\mathcal{D}_2}{\alpha_j, \Gamma_2 \Rightarrow \Delta_2, \alpha_i} (\text{Res})}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, \alpha_i} \quad \frac{\mathcal{D}_3}{\alpha_i, \Gamma_3 \Rightarrow \Delta_3} (\text{Res})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3} (\text{Res})$$

with

$$\frac{\frac{\frac{\mathcal{D}_1}{\Gamma_1 \Rightarrow \Delta_1, \alpha_j, \alpha_i} \quad \frac{\mathcal{D}_3}{\alpha_i, \Gamma_3 \Rightarrow \Delta_3} (\text{Res})}{\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3, \alpha_j} \quad \frac{\frac{\mathcal{D}_2}{\alpha_j, \Gamma_2 \Rightarrow \Delta_2, \alpha_i} \quad \frac{\mathcal{D}_3}{\alpha_i, \Gamma_3 \Rightarrow \Delta_3} (\text{Res})}{\alpha_j, \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3} (\text{Res})}{\Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta_1, \Delta_2, \Delta_3} (\text{Res})$$

If $\Gamma_1, \Gamma_3 \Rightarrow \Delta_1, \Delta_3, \alpha_j$ or $\alpha_j, \Gamma_2, \Gamma_3 \Rightarrow \Delta_2, \Delta_3$ happens to be a critical clause (if Γ_3 or Δ_3 contains α_j), we have to perform the above procedure to eliminate critical clauses. Since this procedure sometimes removes, but never interchanges resolution steps, it has no disturbing effect on the result of our permutation of resolutions.²¹ By successively permuting resolutions and eliminating critical clauses we can obtain a derivation \mathcal{D}' of a subclause κ' of κ'' (and therefore of κ), which shares with \mathcal{D}'' properties (1)–(3), and which in addition has the feature that no α_j -resolution precedes an α_i -resolution if $1 \leq i < j \leq n$. Furthermore, we can assume that no α -resolution for an α occurring in κ' is used in \mathcal{D}' , since otherwise α could be permuted down to be the last resolution step which would eliminate α . This implies that in \mathcal{D}' no α can occur below an α -resolution, because otherwise α would occur in κ' . Therefore, if in \mathcal{D}' an α -resolution is followed by a β -resolution, the β -resolution cannot have a premise in

21 Other cases are treated similarly. In the case presented here, both \mathcal{D}_1 and \mathcal{D}_2 end with sequents containing α_i on the right hand side, i.e. the resolution step applied to \mathcal{D}_1 and \mathcal{D}_2 contains an implicit contraction with respect to α_i . – I owe to Rodrigo Read-Nasser the reference to this point, which I had overlooked in a previous version.

which α occurs. This corresponds to the fact that $Res^{\sigma^* \alpha}(K)$ generates only clauses *without* α .

It is easy to see that these results imply the assertions of Lemma 2.

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