To appear in the Journal of Philosophical Logic

Implications-as-rules vs. implications-as-links: An alternative implication-left schema for the sequent calculus

Peter Schroeder-Heister*
Wilhelm-Schickard-Institut für Informatik
Universität Tübingen
Sand 13, 72076 Tübingen, Germany
psh@informatik.uni-tuebingen.de

Abstract

The interpretation of implications as rules motivates a different left-introduction schema for implication in the sequent calculus, which is conceptually more basic than the implication-left schema proposed by Gentzen. Corresponding to results obtained for systems with higher-level rules, it enjoys the subformula property and cut elimination in a weak form.

The introduction schema for implication on the left side of the turnstile in Gentzen's sequent calculus for intuitionistic logic runs as follows:

$$(\rightarrow \mathrm{L})\ \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \quad .$$

With all other logical inference schemata it shares (among others) the following two properties:

(I) The schema contains exactly one connective.

(II) The connective occurs only in the conclusion of the schema, i.e. below the inference line, and there exactly once.

*This paper was written during a research stay at IHPST Paris supported by the Fondation Maison des Sciences de l'Homme, by the ESF research project "Dialogical Foundations of Semantics (DiFoS)" within the ESF-EUROCORES programme "LogICCC — Modelling Intelligent Interaction" (DFG Schr 275/15-1) and the French-German DFG-ANR project "Hypothetical Reasoning — Logical and Semantical Perspectives (HYPOTHESES)" (DFG Schr 275/16-1). I gratefully acknowledge the comments of participants of a seminar given at IHPST on the subject of this paper.

(III) In the conclusion of the schema, i.e., below the inference line, each formula variable representing an argument of the connective[1] occurs exactly once.

According to property (I) it is not possible to introduce two connectives simultaneously, according to property (II) a connective cannot be introduced simultaneously at different positions in a sequent, and according to property (III) a connective is introduced together with its arguments into a (structural) context, which is completely independent of these arguments.

The sequent calculus for intuitionistic propositional logic[2] based on this schema, in the following called SC, enjoys cut elimination and therefore has the subformula property. A similar result can be obtained for a whole range of logical systems if one just requires certain properties of logical inference schemata to hold including (I) - (III) (see, e.g., [3]).

As an alternative to ($\to$L), we consider the following left-introduction schema for implication:

$$(\to \mathrm{L})^\circ\ \frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}\ .$$

Obviously, this schema satisfies properties (I) and (II), but not (III), as the schematic letter B occurs twice in its conclusion, both as an argument of $\to$ on the left side of the turnstile, and independently of $\to$ on its right side.

It can easily be seen that the schemata ($\to$L)$^\circ$ and ($\to$L) are equivalent in the sense that, given the premisses of ($\to$L), we can infer its conclusion by means of ($\to$L)$^\circ$, and conversely, given the premiss of ($\to$L)$^\circ$, we can infer its conclusion by means of ($\to$L). (Note that we compare ($\to$L)$^\circ$ and ($\to$L) as *schemata*, not as concrete instances.) To derive ($\to$L) from ($\to$L)$^\circ$ we must use cut. To derive ($\to$L)$^\circ$ from ($\to$L) cut is not needed.

The calculus based on ($\to$L)$^\circ$, in the following called SC$^\circ$, lacks the cut elimination property, as the following example demonstrates:

$$(\mathrm{Cut})\ \frac{(\to \mathrm{L})^\circ\ \dfrac{A \vdash A}{A, A \to B \wedge C \vdash B \wedge C} \qquad (\wedge \mathrm{L})\ \dfrac{B \vdash B}{B \wedge C \vdash B}}{A, A \to B \wedge C \vdash B}\ . \tag{1}$$

As there is no inference schema apart from cut that allows one to generate $A, A \to B \wedge C \vdash B$ (if A, B and C are atomic and different), this application of cut cannot be avoided.

---

[1]In ($\to$L), these variables are A and B, but not C (which is a variable for a formula used 'structurally' or 'parametrically', but not as a argument of the connective introduced, similarly to the variables Γ and Δ, which stand for sets, multisets, or lists).

[2]More precisely, positive or minimal logic, as we do not discuss the handling of negation and/or absurdity.

However, SC$^\circ$ still enjoys cut elimination in a weak form and, as a corollary of that, the subformula principle. More precisely, (1) exemplifies the only situation, in which cut fails.

Weak cut elimination for SC$^\circ$:
Every derivation in SC$^\circ$ *(with cut) can be transformed into a derivation, in which cut occurs only in the situation, where its left premiss is the conclusion of* $(\to\mathrm{L})^\circ$*, i.e., in the following situation:*

$$\dfrac{(\to\mathrm{L})^\circ\ \dfrac{\vdots}{\Gamma\vdash A}\qquad \dfrac{\vdots}{\Delta, A\vdash C}}{\Gamma,\Delta\vdash C}\ (\mathrm{Cut})\ . \tag{2}$$

We can even assume that the right premiss $\Delta, A\vdash C$ is either itself a conclusion of an inference figure of form (2), or results from introducing the cut formula A in the last step by means of an L-inference. A cut of the form (2) is needed in order to derive $(\to\mathrm{L})$ from $(\to\mathrm{L})^\circ$. In fact, if we consider a purely implicational system and give $(\to\mathrm{L})^\circ$ the multi-ary form

$$\frac{\Gamma\vdash A_1\quad\ldots\quad\Gamma\vdash A_n}{\Gamma, A_1\to\ (\ldots\ (A_n\to B)\ldots)\vdash B}\ , \tag{3}$$

then we have full cut elimination, as remarked by Avron [1].[3] It is immediately obvious that weak cut elimination implies the subformula principle, as the cut formula A in (2) is contained in an implication $B\to A$ which is introduced by means of $(\to\mathrm{L})^\circ$ and therefore belongs to Γ. The weak cut elimination property for SC$^\circ$ can be extracted from the consideration of natural-deduction calculi with rules of higher levels as investigated in [9, 10] of which SC$^\circ$ is a sequent-style translation ([11, 13]). Alternatively, it can be proved directly. The simplest argument in the present context is the following: We translate the given derivation in SC$^\circ$ into one in SC, then eliminate cuts there, and then translate the resulting derivation in SC back into SC$^\circ$. This backwards translation only creates cuts of the form (2).

Our reason for suggesting $(\to\mathrm{L})^\circ$ is that this schema is conceptually more elementary and more plausible than the standard schema $(\to\mathrm{L})$. It is based on the idea that an implication $A\to B$ expresses a rule, namely the rule which allows one to pass over from A to B. Assuming the rule $A\to B$ gives us a reason to move from the assertion A to the assertion B. This *implications-as-rules* interpretation is a natural way of interpreting implication in natural deduction. In fact, the system SC$^\circ$ can be viewed as a translation of this view from natural deduction into the sequent calculus.

[3] Avron also remarks that $(\to\mathrm{L})$ is a way of avoiding the multi-ary character of this schema, which cannot be effected by means of $(\to\mathrm{L})^\circ$ alone (if conjunction is not available).

In natural deduction, the implications-as-rules view reads modus ponens as application of a rule, i.e.

$$\frac{A \to B \quad A}{B} \quad \text{as} \quad A \to B\,\frac{A}{B}\;. \tag{4}$$

This is made fully explicit in a system with assumption rules, in which rules $A \Rightarrow B$ are distinguished notationally from implications $A \to B$, and in which implications $A \to B$ are formally reduced to rules $A \Rightarrow B$ by explicit introduction and elimination inferences, where the elimination inferences take a generalized form (see [13] and the references therein). As far as standard natural deduction with modus ponens as the $\to$elimination schema is concerned, the implications-as-rules view is an *interpretation* of implication which as its natural sequent-style counterpart has the system SC° with $(\to \mathrm{L})^\circ$ as $\to$left schema.

Even if the subformula property is preserved, giving up full cut elimination in favour of the implications-as-rules view as a conceptually plausible interpretation of implication makes sense only if the interpretation of implication underlying the traditional $(\to \mathrm{L})$ schema is less plausible or less elementary than the implications-as-rules view. If we use (4) as a way of representing modus ponens, we can display the translation of applications of $(\to \mathrm{L})^\circ$ and $(\to \mathrm{L})$ into natural deduction as follows, where $\mathcal{D}_1'$ and $\mathcal{D}_2'$ are natural-deduction translations of the corresponding subderivations $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively[4]:

$$(\to \mathrm{L})^\circ\,\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array}}{\Gamma, A \to B \vdash B} \quad \text{translates into} \quad A \to B\,\frac{\begin{array}{c}\mathcal{D}_1'\\ A\end{array}}{B} \tag{5}$$

$$(\to \mathrm{L})\,\frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \vdash A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \Delta, B \vdash C\end{array}}{\Gamma, \Delta, A \to B \vdash C} \quad \text{translates into} \quad \begin{array}{c} A \to B\,\dfrac{\begin{array}{c}\mathcal{D}_1'\\ A\end{array}}{B} \\ \mathcal{D}_2' \\ C \end{array}\;. \tag{6}$$

Using this representation, in the first case, the application of $A \to B$ is used to continue the derivation $\mathcal{D}_1'$ from A to B, whereas in the second case, the application of $A \to B$ is used to *link together* the derivation $\mathcal{D}_1'$ of A with the derivation $\mathcal{D}_2'$ of C from B. Therefore we may speak of the *implications-as-links* interpretation as underlying the $(\to \mathrm{L})$ schema of the standard sequent calculus SC, in contradistinction to the *implications-as-rules* view underlying the modified system SC°.

This representation makes it also plausible, why in the interpretation-as-links interpretation we have full cut elimination. The situation, in which cut elimination can fail under the implications-as-rules intepretation is a situation, in which the cut formula is

[4]As common in proof theory, formulas or sequents above and below script letters belong (as assumptions or conclusions) to the derivations denoted by the script letters.

a conclusion of an implication introduced in the last step. However, this is exactly the situation, where in the translation (6) the derivations

$$A \to B \,\frac{\stackrel{\mathcal{D}_1'}{A}}{B} \qquad \text{and} \qquad \begin{array}{c} B \\ \mathcal{D}_2' \\ C \end{array}$$

are joined.

From this point of view, the schema $(\to \mathrm{L})$ implicitly contains a form of cut with the cut formula B, since the linking of $\mathcal{D}_1'$ and $\mathcal{D}_2'$ by means of $A \to B$ can be split up into the application of $A \to B$ as a rule yielding B, plus the identification of this B with the assumption B of $\mathcal{D}_2'$, which corresponds to a cut. This cut is made explicit when only the implications-as-rules interpretation is available. This gives the implications-as-rules interpretation a philosophical advantage, as it is more elementary. The implications-as-links interpretation is more complex, as it adds to the implications-as-rules interpretation a weak version of cut.

That some sort of cut is contained in the implications-as-links view can also be seen from the fact that $(\to \mathrm{L})$ can be used to enforce cut by just adding a trivial implication. By introducing $A \to A$ on the left side of the turnstile, we obtain cut with cut formula A:

$$(\to \mathrm{L}) \;\; \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta, A \to A \vdash C}\,. \tag{7}$$

This cannot be achieved by means of $(\to \mathrm{L})^\circ$ without using weak cut (i.e., an inference figure of form (2)). Under the implications-as-rules view the trivial assumption $A \to A$ has no power at all. In the implications-as-links view an implication carries more than the licence to pass from one formula to another.[5]

We have proposed $(\to \mathrm{L})^\circ$ as an alternative left introduction schema for $\to$. More precisely, we have split up Gentzen's schema $(\to \mathrm{L})$ into the more elementary schema $(\to \mathrm{L})^\circ$ plus a weak version of cut, which does not destroy the subformula principle. Admittedly, we relied on plausibility considerations to motivate $(\to \mathrm{L})^\circ$, appealing to the interpretation of implications as rules. This interpretation of implication is not mandatory. However, it is very elementary, given that the notion of a rule is a primitive notion used to describe reasoning, actions and events at a very basic level. To us it is more plausible than the implications-as-links interpretation underlying $(\to \mathrm{L})$. Even though, literally speaking, as speech acts, rules are not statements or sentences, one

[5]This means that in the presence of $(\to \mathrm{L})$, the power of cut (beyond what is already contained in $(\to \mathrm{L})$) only consists in allowing to eliminate trivial assumptions:

$$\frac{\Gamma, A \to A \vdash C}{\Gamma \vdash C}\,.$$

For a discussion of cut in this sense in a λ-calculus setting see [2].

may look at implications as a sort of 'frozen' rules (see [6], p. 78 [English translation p. 70]) that are put into action in a certain way: When a rule is *assumed*, it permits us to move from one statement to another, and, conversely, when we have shown that we may move from one statement to another, we have *established* a rule.

Gentzen himself motivated his sequent calculus essentially by technical considerations, as a means to prove his Hauptsatz (see [4], p. 191 [English translation p. 83]). For that purpose SC is better suited than SC°. However, as there are good reasons to interpret the sequent calculus as a conceptual achievement in its own right (see [12]) rather than only as a technical device or as a sort of 'metacalculus' of natural deduction (see [8], p. 90), it is definitely appropriate to look for philosophically more plausible variants of its inference schemata independently of the question of cut elimination. This holds even more as one of the main corrollaries of cut elimination, the subformula principle, continues to hold for our alternative system, even though full cut elimination is lost. Something similar holds for the principles (I) – (III) mentioned at the beginning of this article. Whereas (I) and (II) can be given good semantic arguments, essentially based on the idea of a stepwise introduction of new constants and the locality of meaning, (III) is by far not so plausible.

We emphasize again that our considerations hold for the intuitionistic system only. For a symmetric system with more than one formula in the succedent, the situation is different. There one can even argue that $(\to \mathrm{L})$ (or the classical schema corresponding to it) is most appropriate (see [14]). However, in this case we might question that $\to$ is a genuine notion of implication at all and not just an operator defined in terms of negation and disjunction or conjunction. In any case it is not comparable to the intuitionistic concept of implication.[6]

[6]We have not discussed features corresponding to $(\to \mathrm{L})^\circ$ and to weak cut elimination in natural deduction. For that one has to discuss generalized elimination inferences, in which the interpretation of implications as rules is appropriately framed (see [13] and the references therein). The investigations reported here have been developed on the background of the implications-as-rules interpretation put forward by the author since 1981, and by carrying over some of their ideas from natural deduction to the sequent calculus (see [1, 11]). Negri and von Plato [7] (p. 184) mention the rule $(\to \mathrm{L})^\circ$ as a sequent calculus rule corresponding to modus ponens, followed by a counterexample to cut analogous to (1), which is based on implication only. This counterexample shows again that for cut elimination in the implicational system the multi-ary form (3) of $(\to \mathrm{L})^\circ$ considered in [1] and the corresponding forms of rule introduction in the antecedent considered in [11] and [5] are really needed. From a different point of view, Tesconi [15] considers a sequent calculus with a restricted form of cut, which is different from our weak cut in the sense of (2), but which also satisfies the subformula principle. Work in progress by Jean Fichot and the author investigates the precise relationship between standard natural deduction and SC° in a λ-calculus setting.

References

[1] Arnon Avron. Gentzenizing Schroeder-Heister's natural extension of natural deduction. *Notre Dame Journal of Formal Logic*, 31:127–135, 1990.

[2] Henk Barendregt and Silvia Ghilezan. Lambda terms for natural deduction, sequent calculus and cut elimination. *Journal of Functional Programming*, 10:121–134, 2000.

[3] Nuel D. Belnap. Display logic. *Journal of Philosophical Logic*, 11:375–417, 1982.

[4] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431 (English translation in: *The Collected Papers of Gerhard Gentzen* (ed. M. E. Szabo), Amsterdam: North Holland (1969), pp. 68–131), 1934/35.

[5] Lars Hallnäs and Peter Schroeder-Heister. A proof-theoretic approach to logic programming: I. Clauses as rules. II. Programs as definitions. *Journal of Logic and Computation*, 1:261–283, 635–660, 1990/91.

[6] Gisbert Hasenjaeger. *Einführung in die Grundbegriffe und Probleme der modernen Logik* (English Translation: *Introduction to the Basic Concepts and Problems of Modern Logic,* Reidel, Dordrecht, 1972). Alber, Freiburg, 1962.

[7] Sara Negri and Jan von Plato. *Structural Proof Theory.* Cambridge University Press, 2001.

[8] Dag Prawitz. *Natural Deduction: A Proof-Theoretical Study.* Almqvist & Wiksell (Reprinted Mineola NY: Dover Publ., 2006), Stockholm, 1965.

[9] Peter Schroeder-Heister. *Untersuchungen zur regellogischen Deutung von Aussagenverknüpfungen.* Doctoral dissertation [see author's homepage], Universität Bonn, 1981.

[10] Peter Schroeder-Heister. A natural extension of natural deduction. *Journal of Symbolic Logic*, 49:1284–1300, 1984.

[11] Peter Schroeder-Heister. *Structural Frameworks with Higher-Level Rules.* Habil. thesis [see author's homepage], Universität Konstanz, 1987.

[12] Peter Schroeder-Heister. Sequent calculi and bidirectional natural deduction: On the proper basis of proof-theoretic semantics. In M. Peliš, editor, *The Logica Yearbook 2008.* College Publications, London, 2009.

[13] Peter Schroeder-Heister. Generalized elimination inferences, higher-level rules, and the implications-as-rules interpretation of the sequent calculus. In Edward Hermann Haeusler, Luiz Carlos Pereira, and Valeria de Paiva, editors, *Advances in Natural Deduction*. 2010.

[14] Peter Schroeder-Heister. Definitional reflection and Basic Logic. *Annals of Pure and Applied Logic (Special issue, Festschrift 60th Birthday Giovanni Sambin)*, (submitted for publication).

[15] Laura Tesconi. Some not so obvious remarks about the cut rule. In Carlo Marletti, editor, *First Pisa Colloquium in Logic, Language and Epistemology*. ETS, Pisa, 2010.