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Generalized elimination inferences, higher-level rules, and the implications-as-rules interpretation of the sequent calculus

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Abstract

We investigate the significance of higher-level generalized elimination rules as proposed by the author in relation to standard-level generalized elimination rules as proposed by Dyckhoff, Tennant, López-Escobar and von Plato. Many of the results established for natural deduction with higher-level rules such as normalization and the subformula principle immediately translate to the standard-level case. The sequent-style interpretation of higher-level natural deduction as proposed by Avron and by the author leads to a system with a weak rule of cut, which enjoys the subformula property. The interpretation of implications as rules motivates a different left-introduction schema for implication in the ordinary (standard-level) sequent calculus, which conceptually is more basic than the implication-left schema proposed by Gentzen. Corresponding to the result for the higher-level system, it enjoys the subformula property and cut elimination in a weak form.

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1 Generalized higher-level elimination rules

In Schroeder-Heister [24, 26], generalized elimination rules for logical constants were proposed in order to obtain a general schema for introduction and elimination rules for propositional operators[1]. Given m introduction rules for an n-ary constant of propositional logic c

$$(c\,\mathrm{I})\ \ \frac{\Delta_1(A_1,\dots,A_n)}{c(A_1,\dots,A_n)}\ \ \dots\ \ \frac{\Delta_m(A_1,\dots,A_n)}{c(A_1,\dots,A_n)}\ ,$$

where the $\Delta_i(A_1,\dots,A_n)$ are premisses structured in a certain way, the elimination rule is

$$(c\,\mathrm{E})\ \ \frac{c(A_1,\dots,A_n)\qquad \begin{matrix}[\Delta_1(A_1,\dots,A_n)]\\ C\end{matrix}\quad \dots \quad \begin{matrix}[\Delta_m(A_1,\dots,A_n)]\\ C\end{matrix}}{C}\ ,$$

where the brackets indicate the possibility of discharging the assumption structures mentioned. For conjunction and implication these rules specialize to the following:

$$(\wedge\,\mathrm{I})\ \frac{A\quad B}{A\wedge B}\qquad (\wedge\,\mathrm{E_{GEN}})\ \frac{A\wedge B\qquad \begin{matrix}[A\quad B]\\ C\end{matrix}}{C}\qquad (\rightarrow\mathrm{I})\ \frac{\begin{matrix}[A]\\ B\end{matrix}}{A\rightarrow B}\qquad (\rightarrow\mathrm{E_{HL}})\ \frac{A\rightarrow B\qquad \begin{matrix}[A\Rightarrow B]\\ C\end{matrix}}{C}\ .$$

Here $(\wedge\,\mathrm{I})$ and $(\rightarrow\mathrm{I})$ are the usual introduction rules for conjunction and implication, and $(\wedge\,\mathrm{E_{GEN}})$ and $(\rightarrow\mathrm{E_{HL}})$ are their generalized elimination rules, following the pattern of $(c\,\mathrm{E})$. The index "HL" indicates that our generalized elimination schema for $\rightarrow$ is a rule of higher level, in contradistinction to generalized standard-level elimination rules, which will be discussed in section 2.

The motivation for the generalized introduction and elimination rules $(c\,\mathrm{I})$ and $(c\,\mathrm{E})$ can be given in different ways: One is that the introduction rules represent a kind of 'definition' of c, and the elimination rule says that everything that follows from each defining condition $\Delta_i(A_1,\dots,A_n)$ of $c(A_1,\dots,A_n)$ follows from $c(A_1,\dots,A_n)$ itself. This can be seen as following Gentzen's ([6] p. 189) consideration of introduction inferences in natural deduction as a sort of definition, or as applying an *inversion principle* or a principle of *definitional reflection* to the introduction rules (for Lorenzen's 'inversion principle' and the related principle of definitional reflection and its usage in proof-theoretic semantics, see [8, 29, 31, 32]).

A slightly different idea (preferred by the author in [24, 26]) that focusses more on the elimination rules is that $c(A_1,\dots,A_n)$ expresses the 'common content' of

[1] In [25] this approach was extended to quantifiers, and in [27] to various other applications, including the sequent calculus, certain substructural logics (in particular relevant logic), and Martin-Löf's logic. In [7, 29] and in later publications it was extended to the realm of clausal based reasoning in general.

$\Delta_1(A_1, \ldots, A_n), \ldots, \Delta_m(A_1, \ldots, A_n)$, i.e. the set of consequences of $c(A_1, \ldots, A_n)$ is exactly the intersection of the sets of consequences of the $\Delta_i(A_1, \ldots, A_n)$ $(1 \leq i \leq n)$. Formally, both interpretations amount to the same, viz. the 'indirect' form of the elimination rule which generalizes the standard pattern of introduction and elimination rules for disjunction:

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad \frac{A{\vee}B \quad \begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{C} \; .$$

In the generalized elimination rule $(c\,\mathrm{E})$ the premiss structures occur in assumption position. In the case of $\rightarrow$ this means that the dependency of B from A must be represented as a form of assumption. The proposal of Schroeder-Heister [24, 26] was to read this dependency as a *rule*, since the fact that B has been derived from A can be naturally understood as the fact that the rule allowing one to pass over from A to B has been established. Conversely, assuming that the rule permitting to pass from A to B is available, is naturally understood as expressing that we can in fact derive B from A. Consequently, the minor premiss of generalized $\rightarrow$ elimination $(\rightarrow \mathrm{E}_{\mathrm{HL}})$ depends on the rule $A \Rightarrow B$, which is discharged at the application of this elimination inference. $(\rightarrow \mathrm{E}_{\mathrm{HL}})$ is therefore a rule of higher level, i.e., a rule that allows for a proper rule to be discharged, rather than for only formulas as (dischargeable) assumptions. Actually, the 'usual' $\rightarrow$ elimination rule, which is modus ponens, is a rule without any discharge of assumptions at all:

$$(\rightarrow \mathrm{E}_{\mathrm{MP}})\ \frac{A \rightarrow B \quad A}{B}\ ,$$

as are the usual $\wedge$ elimination rules:

$$(\wedge\,\mathrm{E})\ \frac{A{\wedge}B}{A} \qquad \frac{A{\wedge}B}{B}\ .$$[2]

This approach is formalized in a calculus of rules of higher levels: A rule of level 1 is a formula, and a rule of level $\ell + 1$ has the form $(R_1, \ldots, R_n) \ \Rightarrow \mathrm{A}$, where the

[2] A generalized schema for elimination rules which contains $(\wedge\,\mathrm{E}_{\mathrm{GEN}})$ as an instance was proposed by Prawitz [23]. I have made this point clear in all relevant publications. In fact, a fundamental error in Prawitz's treatment of implication in his elimination schema was one of my motivations to develop the idea of rules of higher levels, together with Kutschera's [11] treatment of logical constants in terms of an iterated sequent arrow in what he very appropriately called *Gentzen semantics* (Ger. 'Gentzensemantik'). So I claim authorship for the general schema for elimination rules, but not for the particular idea of generalized conjunction elimination $(\wedge\,\mathrm{E}_{\mathrm{GEN}})$. I mention this point as I have been frequently acknowledged as the author or $(\wedge\,\mathrm{E}_{\mathrm{GEN}})$, without reference to its embedding into a higher-level framework. When the higher-level framework is mentioned, it is often not considered relevant, as transcending the means of expression of standard natural deduction. Although it extends natural deduction, the results obtained for the general higher-level case can easily be specialized to the case of generalized standard-level elimination rules for implication. In this sense, a direct normalization and subformula proof for standard-level systems with generalized rules is already contained in [24]. — The

premisses $R_1, \ldots, R_n$ are rules of maximum level ℓ and the conclusion A is a formula. A finite list of rules is denoted by a capital Greek letter, so in general a rule has the form $\Delta \Rightarrow A$.

A rule can be applied and discharged in the following way: A rule of level 1 works like an axiom, a rule of level 2 generates its conclusion from its premisses:

$$A_1, \ldots, A_n \Rightarrow B \quad \frac{A_1 \quad \ldots \quad A_n}{B}$$

and a rule of level ≥ 3 of the form

$$(\Delta_1 \Rightarrow A_1, \; \ldots \; , \Delta_n \Rightarrow A_n) \;\Rightarrow\; B$$

generates B from $A_1, \ldots, A_n$, whereby the assumptions $\Delta_1, \ldots, \Delta_n$ can be discharged at this application:

$$(\Delta_1 \Rightarrow A_1, \; \ldots \; , \Delta_n \Rightarrow A_n) \Rightarrow B \quad \frac{\begin{matrix}[\Delta_1] & & [\Delta_n] \\ A_1 & \ldots & A_n\end{matrix}}{B} \; . \tag{1}$$

Applying $\Delta \Rightarrow A$ means at the same time assuming it, i.e., introducing it as an assumption except in the case where it is a basic rule (or 'primitive rule') such as $(\rightarrow \mathrm{I})$. Therefore, formally, applying $(\rightarrow \mathrm{I})$ means the same as applying the rule $(A \Rightarrow B) \;\Rightarrow\; A \rightarrow B$:

$$(A \Rightarrow B) \Rightarrow A \rightarrow B \; \frac{\begin{matrix}[A] \\ B\end{matrix}}{A \rightarrow B} \; . \tag{2}$$

However, as in this case $(\rightarrow \mathrm{I})$ is not introduced as an assumption and therefore not used as an *assumption rule*, but is basic (or 'primitive'), we write as usual

$$(\rightarrow \mathrm{I}) \; \frac{\begin{matrix}[A] \\ B\end{matrix}}{A \rightarrow B} \; .$$

The counting of rule levels is according to the height of the tree it corresponds to. For example, in (2) the application of the level-3-rule $(A \Rightarrow B) \;\Rightarrow\; A \rightarrow B$ corresponds to a tree

$$\frac{\begin{matrix}[A] \\ B\end{matrix}}{A \rightarrow B}$$

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phenomenon that acknowledgement is not transitive (A acknowledges exclusively B for something, for which B has explicitly acknowledged C) is a frequent phenomenon. Another example is the *inversion principle* used by Prawitz to philosophically explain the relationship between introduction and elimination inferences in natural deduction. Although Prawitz [21] is fully aware of the fact that he is adopting and adapting a term coined by Lorenzen [13, 14] and explicitly acknowledges its origin, many recent discussions and reformulations of the inversion principle (for example by Negri and von Plato [17]) are apparently not aware of it. For a discussion of inversion in relation to Lorenzen's and Prawitz's semantical approaches see [16, 31, 32].

of height 3. In this sense, the primitive rules $(\to \mathrm{I})$ and $(\vee \mathrm{E})$ are rules of level 3. Obviously, a rule discharging a rule of level 2, i.e. discharging a rule of the form $A_1, \ldots, A_n \Rightarrow B$, must be at least of level 4. The primitive rules used in standard natural deduction are therefore of maximum level 3. A rule of maximum level 3 is also called a *standard-level rule*, whereas a rule of level ≥ 4 is called a proper *higher-level rule*. Thus $(\to \mathrm{E}_{\mathrm{HL}})$ is a proper higher-level (viz., level-4) rule. Obviously, if ℓ is the level of an introduction rule for a logical constant c of the form $(c\ \mathrm{I})$, then $\ell + 1$ is the level of the corresponding elimination rule of the form $(c\ \mathrm{E})$. This raising of levels made it necessary to introduce the concept of higher-level rules and the idea of assuming and discharging rules. This idea can be generalized from the realm of logical constants to arbitrary clausal definitions in logic-programming style and therefore to inductive definitions, leading to a powerful principle of 'definitional reflection' that extends standard ways of dealing with clausal definitions (see [7, 29, 8]).

That modus ponens $(\to \mathrm{E}_{\mathrm{MP}})$ and $(\to \mathrm{E}_{\mathrm{HL}})$ are equivalent can be seen as follows: Suppose $(\to \mathrm{E}_{\mathrm{MP}})$ is a primitive rule. Suppose we have derived the premisses of $(\to \mathrm{E}_{\mathrm{HL}})$, i.e. we have a derivation $\begin{matrix} \mathcal{D}_1 \\ A \to B \end{matrix}$ of $A \to B$ and a derivation $\begin{matrix} A \Rightarrow B \\ \mathcal{D}_2 \\ C \end{matrix}$ of C from the assumption rule $A \Rightarrow B$. If we replace every application of the assumption rule $A \Rightarrow B$ in $\mathcal{D}_2$

$$A \Rightarrow B\ \dfrac{\begin{matrix} \mathcal{D}_3 \\ A \end{matrix}}{B}$$

with an application of $(\to \mathrm{E}_{\mathrm{MP}})$ using $\mathcal{D}_1$ as derivation of its major premiss

$$(\to \mathrm{E}_{\mathrm{MP}})\ \dfrac{\begin{matrix} \mathcal{D}_1 \\ A \to B \end{matrix} \qquad \begin{matrix} \mathcal{D}_3 \\ A \end{matrix}}{B} \quad ,$$

then we obtain a derivation of C, i.e. of the conclusion of $(\to \mathrm{E}_{\mathrm{HL}})$. Note that this derivation of $(\to \mathrm{E}_{\mathrm{HL}})$ from $(\to \mathrm{E}_{\mathrm{MP}})$ works for every single rule instance, i.e., for A, B, C fixed, so we need not assume that $(\to \mathrm{E}_{\mathrm{HL}})$ and $(\to \mathrm{E}_{\mathrm{MP}})$ are rule schemata.

Conversely suppose $(\to \mathrm{E}_{\mathrm{HL}})$ is a primitive rule. Suppose we have derived the premisses of $(\to \mathrm{E}_{\mathrm{MP}})$, i.e. we have derivations $\begin{matrix} \mathcal{D}_1 \\ A \to B \end{matrix}$ and $\begin{matrix} \mathcal{D}_2 \\ A \end{matrix}$ of $A \to B$ and A respectively. From $\mathcal{D}_2$, using the assumption rule $A \Rightarrow B$, we obtain

$$A \Rightarrow B\ \dfrac{\begin{matrix} \mathcal{D}_2 \\ A \end{matrix}}{B} \quad ,$$

from which, together with $\mathcal{D}_1$ we obtain by means of $(\to \mathrm{E}_{\mathrm{HL}})$

$$\dfrac{\begin{matrix} \mathcal{D}_1 \\ A \to B \end{matrix} \qquad {}^1[A \Rightarrow B]\ \dfrac{\begin{matrix} \mathcal{D}_2 \\ A \end{matrix}}{B}\,{}_1}{B} \quad ,$$

which is a derivation of the conclusion B of ($\to$ E$_{\mathrm{MP}}$). However, it is important to note that unlike the other direction, the derivation of ($\to$ E$_{\mathrm{MP}}$) from ($\to$ E$_{\mathrm{HL}}$) does not work for every single rule instance, i.e., for A, B, C arbitrarily fixed. Rather, we must be able to substitute B for C. So what we have essentially proved is that ($\to$ E$_{\mathrm{HL}}$), as a *rule schema*, implies ($\to$ E$_{\mathrm{MP}}$).

This corresponds to the original idea of generalized elimination rules of the form (c E), where the C is implicitly universally quantified, expressing that *every* C that can be derived from each defining condition of $c(A_1, \ldots, A_n)$ (for fixed $A_1, \ldots, A_n$), can be derived from $c(A_1, \ldots, A_n)$ itself.

We have an entirely similar situation when proving the equivalence of ($\wedge$ E) and ($\wedge$ E$_{\mathrm{GEN}}$). For the direction from left to right, given derivations $\begin{array}{c}\mathcal{D}_1\\ A\wedge B\end{array}$ and $\begin{array}{c}A\quad B\\ \mathcal{D}_2\\ C\end{array}$, we construct a derivation

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\wedge B\end{array}}{A}\quad\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\wedge B\end{array}}{B}\\ \mathcal{D}_2\\ C\end{array},$$

whereas for the direction from right to left we instantiate C with A and B respectively, yielding derivations

$$\dfrac{A\wedge B \qquad {}^{1}[A]}{A}\,{}_{1} \quad \text{and} \quad \dfrac{A\wedge B \qquad {}^{1}[B]}{B}\,{}_{1},$$

where $[A]$ represents a derivation of A from A and B, in which A is effectively discharged and B is vacuously discharged, and analogously with $[B]$.

A detailed proof of normalization and of the subformula property for higher-level natural deduction, for systems with and without the absurdity rule (the latter corresponding to minimal logic), and for a system with explicit denial can be found in [24][3]. Generalized elimination rules have also found their way into type-theoretical approaches such as Martin-Löf's type theory (see [15], p. i–iii), where they are particularly useful in the treatment of dependent products (see [5]), uniform approaches to the machine-assisted treatment of logical systems such as the Edinburgh logical framework (see [9]), proof-editors and theorem provers such as Isabelle ([18]) and others. For applications of this framework to relevance logic, logic programming and Martin-Löf-style logical systems see [27].

In this paper, we confine ourselves to minimal logic, or more precisely, to what corresponds to minimal logic in the higher-level framework, namely nonempty systems of rules and n-ary operators definable from them. Especially we do not touch here the

[3]Although stated as a result, the proof was omitted from the journal publication [26], both for space constraints and because its character is that of an exercise.

problem of dealing with empty systems of rules (which leads to intuitionistic absurdity) or the general problem of denial and negation. Also, concerning substructural issues, we just assume the standard provisions according to which lists of formulas or rules can be handled as sets, where appropriate. In particularly we do not discuss the issue of relevance, although the higher-level framework can be adapted to these fields (see [27, 28]). (Tennant's [35] approach to generalized elimination inferences is much inspired by the problems of relevant implication.) The vast area of using (often extended) structural means of expression to describe or determine the inferential meaning of logical constants, which in particular leads to general frameworks for describing logics is also omitted from our discussion here. For this point see Wansing [38].

2 Generalized standard-level elimination rules

Dyckhoff [4] (in 1988), Tennant [35] (in 1992), López-Escobar [12] (in 1999), and von Plato [19, 20, 17] (2001-2002) and Tennant [36] (in 2002) have independently (though, as is clear, with considerable temporal intervals) proposed a generalized form of the $\rightarrow$ elimination rule which is related to the left $\rightarrow$ introduction rule in sequent calculi. In our terminology it is of level 3 and thus a standard-level rule. We call it $(\rightarrow \mathrm{E}_{\mathrm{SL}})$:

$$(\rightarrow \mathrm{E}_{\mathrm{SL}})\ \frac{A \rightarrow B \quad A \quad \begin{matrix}[B]\\ C\end{matrix}}{C}\ .$$[4]

In these approaches the generalized elimination rule for conjunction is the same as before: $(\wedge\, \mathrm{E}_{\mathrm{GEN}})$, which means that it is a level-3-rule and thus one level higher than the $\wedge$ introduction rules. However, $(\rightarrow \mathrm{E}_{\mathrm{SL}})$ stays at the same level as $(\rightarrow \mathrm{I})$ (viz. level 3), so that no higher level is needed as far as the standard connectives are concerned. As these approaches are concerned with generalizing the elimination rules of the standard connectives and not with a general schema for arbitrary n-ary connectives, they are content with $(\wedge\, \mathrm{E}_{\mathrm{GEN}})$ and $(\rightarrow \mathrm{E}_{\mathrm{SL}})$. $(\wedge\, \mathrm{E}_{\mathrm{GEN}})$ and $(\rightarrow \mathrm{E}_{\mathrm{SL}})$ are generalized elimination rules, as the indirect pattern of elimination found in $(\vee\, \mathrm{E})$ is carried over to conjunction and implication, in the sense that the right minor premiss C is repeated in the

[4]Dyckhoff's paper [4] appeared in a volume of a kind often called a 'grey' publication (similar to a technical report), which was difficult to notice and to get hold of, at least in pre-internet times. It mentions $(\rightarrow \mathrm{E}_{\mathrm{SL}})$ neither in the title nor in the abstract nor in the display of the inference rules of natural deduction. The only sentence mentioning $(\rightarrow \mathrm{E}_{\mathrm{SL}})$ explicitly as a possible primitive rule for implication elimination occurs in the discussion of proof tactics for implication: 'this makes it clear that a possible form of the rule $\supset$*_elim* might be that from proofs of $A \supset B$, of A, and (assuming B) of C one can construct a proof of C. Such a form fits in better with the pattern for elimination rules now increasingly regarded as orthodox, and is clearer than the other possibility for $\supset$*_elim* advocated by Schroeder-Heister and Martin-Löf [...]' ([4], p. 55). Before the appearance of von Plato's papers, Dyckhoff never referred to this publication in connection with the particular form of $\rightarrow$ elimination, in order to make his idea visible.

conclusion, even though, due to the presence of the minor premiss A, the uniformity inherent in the generalized higher-level rules is lost in $(\to \mathrm{E_{SL}})$.

We do not discuss here the somewhat different conceptual intentions the above authors pursue when introducing $(\to \mathrm{E_{SL}})$. We are solely interested in the form and power of this rule that we shall compare with $(\to \mathrm{E_{HL}})$ and with the higher-level approach in general. Concerning terminology, we use the term 'generalized elimination rules' as a generic term covering both the higher-level and the standard-level versions. Tennant speaks of the 'parallel form' of elimination rules and of 'parallelized' natural deductions, von Plato of 'general elimination rules'.

It can easily be shown that $(\to \mathrm{E_{MP}})$ and $(\to \mathrm{E_{SL}})$ are equivalent as primitive rules. Suppose $(\to \mathrm{E_{MP}})$ is a primitive rule. Suppose we have derived the premisses of $(\to \mathrm{E_{SL}})$, i.e. we have a derivation $\begin{array}{c}\mathcal{D}_1\\ A\to B\end{array}$ of $A\to B$, a derivation $\begin{array}{c}\mathcal{D}_2\\ A\end{array}$ of A and a derivation $\begin{array}{c}B\\ \mathcal{D}_3\\ C\end{array}$ of C from B. If we replace every occurrence of the assumption B in $\mathcal{D}_3$ with an application of $(\to \mathrm{E_{MP}})$ using $\mathcal{D}_1$ and $\mathcal{D}_2$ as premiss derivations:

$$(\to \mathrm{E_{MP}})\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\to B\end{array}\quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}}{B}\ ,$$

we obtain a derivation of C, i.e. the conclusion of $(\to \mathrm{E_{SL}})$. This derivation of $(\to \mathrm{E_{SL}})$ from $(\to \mathrm{E_{MP}})$ works for every single rule instance, i.e., for A, B, C fixed. Thus we need not assume that $(\to \mathrm{E_{MP}})$ holds as a rule schema.

Conversely suppose $(\to \mathrm{E_{SL}})$ is a primitive rule. Suppose we have derived the premisses of $(\to \mathrm{E_{MP}})$, i.e. we have derivations $\begin{array}{c}\mathcal{D}_1\\ A\to B\end{array}$ and $\begin{array}{c}\mathcal{D}_2\\ A\end{array}$ of $A\to B$ and A, respectively. Considering B to be a derivation of itself, by an application of $(\to \mathrm{E_{SL}})$ we obtain

$$(\to \mathrm{E_{SL}})\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\to B\end{array}\quad \begin{array}{c}\mathcal{D}_2\\ A\end{array}\quad {}^{1}[B]}{B}\,{}_{1}\ ,$$

which is a derivation of the conclusion B of $(\to \mathrm{E_{MP}})$. As before, it is important to note that unlike the other direction, the derivation of $(\to \mathrm{E_{MP}})$ from $(\to \mathrm{E_{SL}})$ does not work for every single rule instance, i.e., for A, B, C arbitrarily fixed. Rather, we must be able to substitute B for C. So what we have essentially proved is that $(\to \mathrm{E_{SL}})$, as a *rule schema*, implies $(\to \mathrm{E_{MP}})$.

This again corresponds to the idea of generalized elimination as an indirect schema, in which the C is implicitly quantified.

3 Comparison of the higher-level with the standard-level generalized $\rightarrow$ elimination rules

As rule schemata, or more precisely, with the letter C understood schematically, both $(\rightarrow \mathrm{E_{HL}})$ and $(\rightarrow \mathrm{E_{SL}})$ are equivalent to $(\rightarrow \mathrm{E_{MP}})$, which implies that $(\rightarrow \mathrm{E_{HL}})$ and $(\rightarrow \mathrm{E_{SL}})$ are equivalent. If we compare $(\rightarrow \mathrm{E_{HL}})$ and $(\rightarrow \mathrm{E_{SL}})$ directly, i.e., not via $(\rightarrow \mathrm{E_{MP}})$, then $(\rightarrow \mathrm{E_{HL}})$ implies $(\rightarrow \mathrm{E_{SL}})$ as a concrete rule, i.e. as an instance with A, B, C fixed, which can be seen as follows. Suppose we have derived the premisses of $(\rightarrow \mathrm{E_{SL}})$, i.e. we have a derivation $\begin{array}{c}\mathcal{D}_1\\ A\rightarrow B\end{array}$ of $A\rightarrow B$, a derivation $\begin{array}{c}\mathcal{D}_2\\ A\end{array}$ of A and a derivation $\begin{array}{c}B\\ \mathcal{D}_3\\ C\end{array}$ of C from B. If we replace every occurrence of the assumption B in $\mathcal{D}_3$ with the following application of the assumption rule $A \Rightarrow B$

$$A \Rightarrow B\,\begin{array}{c}\mathcal{D}_2\\ \dfrac{A}{B}\end{array}\quad ,$$

we obtain a derivation

$$A \Rightarrow B\,\begin{array}{c}\mathcal{D}_2\\ \dfrac{A}{B}\\ \mathcal{D}_3\\ C\end{array}$$

of C from $A \Rightarrow B$. By application of $(\rightarrow \mathrm{E_{HL}})$ we can discharge $A \Rightarrow B$:

$$(\rightarrow \mathrm{E_{HL}})\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\rightarrow B\end{array} \qquad \begin{array}{c}{}^{1}[A \Rightarrow B]\,\begin{array}{c}\mathcal{D}_2\\ \dfrac{A}{B}\end{array}\\ \mathcal{D}_3\\ C\end{array}}{C}\,{}_{1}$$

yielding the conclusion C of $(\rightarrow \mathrm{E_{SL}})$.

For the converse direction, we have to keep C schematic, more precisely, we have to be able to specify it as B. Suppose we have derived the premisses of $(\rightarrow \mathrm{E_{HL}})$, i.e. we have a derivation $\begin{array}{c}\mathcal{D}_1\\ A\rightarrow B\end{array}$ of $A\rightarrow B$ and a derivation $\begin{array}{c}A \Rightarrow B\\ \mathcal{D}_2\\ C\end{array}$ of C from the assumption rule $A \Rightarrow B$. Suppose $A \Rightarrow B$ is actually used as an assumption in $\mathcal{D}_2$ (otherwise $\mathcal{D}_2$ is already the desired derivation of C). If we replace every application of the assumption rule $A \Rightarrow B$ in $\mathcal{D}_2$

$$A \Rightarrow B\,\begin{array}{c}\mathcal{D}_3\\ \dfrac{A}{B}\end{array}$$

with the following application of $(\rightarrow \mathrm{E_{SL}})$

$$(\rightarrow \mathrm{E_{SL}})\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ A\rightarrow B\end{array} \quad \begin{array}{c}\mathcal{D}_3\\ A\end{array} \quad {}^{1}[B]}{B}\,{}_{1}\quad , \tag{3}$$

then we obtain a derivation of C, i.e. of the conclusion of $(\rightarrow \mathrm{E_{HL}})$. The fact that in the application of $(\rightarrow \mathrm{E_{SL}})$ in (3), by instantiating C to B we are using a trivialized version of $(\rightarrow \mathrm{E_{SL}})$, which essentially is modus ponens $(\rightarrow \mathrm{E_{MP}})$, cannot be avoided. It is tempting to consider replacing every application of the assumption rule $A \Rightarrow B$ in $\mathcal{D}_2$

$$\begin{array}{c} \mathcal{D}_3 \\ A \Rightarrow B\ \dfrac{A}{B} \\ \mathcal{D}_4 \\ C \end{array} \tag{4}$$

with the following application of $(\rightarrow \mathrm{E_{SL}})$:

$$(\rightarrow \mathrm{E_{SL}})\ \dfrac{\begin{array}{c}\mathcal{D}_1 \\ A \rightarrow B\end{array} \quad \begin{array}{c}\mathcal{D}_3 \\ A\end{array} \quad \begin{array}{c}{}^1[B] \\ \mathcal{D}_4 \\ C\end{array}}{C}\,1 \tag{5}$$

which would leave the C uninstantiated. However, this way is not viable as in (4) there may occur assumptions open in $\mathcal{D}_3$ but discharged at a rule application in $\mathcal{D}_4$. Such an assumption would remain open in (5), where the derivation (4) is split into two independent parts.

Therefore $(\rightarrow \mathrm{E_{HL}})$ and $(\rightarrow \mathrm{E_{SL}})$ are equivalent with schematic C, but they are not equivalent as instances, i.e. for every particular C. In this sense $(\rightarrow \mathrm{E_{HL}})$ is stronger than $(\rightarrow \mathrm{E_{SL}})$. However, one should not overestimate this difference in strength, as primitive rules of inference are normally understood as rule schemata.[5]

It is easy to see that the mutual translation between $(\rightarrow \mathrm{E_{HL}})$ and $(\rightarrow \mathrm{E_{SL}})$ does not introduce new maximum formulas. Therefore from the normalization and subformula theorems proved in [24] for the generalized higher-level case we immediately obtain the corresponding theorems for generalized standard-level natural deduction. In fact, when specialized from the n-ary case to the case of the standard operators, this proof gives the same reductions as those found in Tennant's [35, 36], López-Escobar's [12] and Negri and von Plato's [17] work. This means there has been an earlier direct proof of normalization for generalized natural deduction than the one given in 1992 by Tennant [35] (there for minimal logic and for Tennant's system of intuitionistic relevant logic), albeit in a more general setting.[6]

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[5]The way in which letters C are schematic, i.e., which propositions may be substituted for them, becomes important, if one investigates extensions of the language considered, i.e., in connection with the problem of uniqueness of connectives (see [3]).

[6]López-Escobar proves *strong* normalization. Other such proofs are by Joachimski & Matthes [10] and by Tesconi [37].

4 Comparison of the expressive power of higher-level with that of standard-level rules

So far we have investigated the relationship between the generalized higher-level $\rightarrow$ elimination rule ($\rightarrow\mathrm{E}_{\mathrm{HL}}$) and the generalized standard-level $\rightarrow$ elimination rule ($\rightarrow\mathrm{E}_{\mathrm{SL}}$). We can carry over some of these results to a more general comparison which relates higher-level and standard-level rules independently of logical connectives. We may ask: Is it possible to express higher-level rules by means of standard-level rules, perhaps along the lines of the comparison of generalized $\rightarrow$ elimination rules in section 3? Here we must again distinguish of whether the rules compared are schematic with respect to some or all propositional letters or not.

The comparison of ($\rightarrow\mathrm{E}_{\mathrm{HL}}$) and ($\rightarrow\mathrm{E}_{\mathrm{SL}}$) might suggest that the rules

$$\frac{\begin{matrix}[A \Rightarrow B]\\ C\end{matrix}}{D} \quad \text{and} \quad \frac{A \quad \begin{matrix}[B]\\ C\end{matrix}}{D} \tag{6}$$

are equivalent. Corresponding to what was shown above, the left rule implies the right one for any fixed A, B, C, D: Given derivations $\begin{matrix}\mathcal{D}_1\\ A\end{matrix}$ and $\begin{matrix}B\\ \mathcal{D}_2\\ C\end{matrix}$ of the premisses of the right rule, we just need to replace each assumption B in $\mathcal{D}_2$ with a step

$$A \Rightarrow B\ \frac{\begin{matrix}\mathcal{D}_1\\ A\end{matrix}}{B}$$

in order to obtain a derivation of the premiss of the left rule. However, the converse direction is not valid: Already if in the left rule of (6) C does not depend on any assumption (i.e., the discharging of $A \Rightarrow B$ is vacuous), the right rule of (6) cannot be obtained, as there is no possibility of generating a derivation of A from the premiss derivations of the left rule. Therefore in (6) the higher-level rule is strictly stronger than the standard-level rule.

If we change D to C, which is more in the spirit of the generalized $\rightarrow$ elimination rule, do we then obtain the equivalence of

$$\frac{\begin{matrix}[A \Rightarrow B]\\ C\end{matrix}}{C} \quad \text{and} \quad \frac{A \quad \begin{matrix}[B]\\ C\end{matrix}}{C} \quad ? \tag{7}$$

As before, the left side implies the right one. However, the right side does not imply the left one, if C is fixed. Only if we allow for C to be substituted with B, which essentially means considering C to be schematic, implies the right side of (7) the left side. By means of

$$\frac{A \quad {}^{1}[B]}{B}\,{}_{1} \quad ,$$

which is the same as

$$\frac{A}{B} \ ,$$

we can eliminate every application of $A \Rightarrow B$ in a given derivation of C from $A \Rightarrow B$, yielding a derivation of C as required. As in the case described in section 3, attempting to extract, from

$$\begin{array}{c} \\ A \Rightarrow B \end{array} \begin{array}{c} \mathcal{D}_1 \\ \frac{A}{B} \\ \mathcal{D}_2 \\ C \end{array} \ ,$$

two independent derivations $\begin{array}{c}\mathcal{D}_1\\ A\end{array}$ and $\begin{array}{c}B\\ \mathcal{D}_2\\ C\end{array}$, is bound to fail, as in $\mathcal{D}_2$ assumptions open in $\mathcal{D}_1$ might have been discharged.

Of course, these are just examples showing that a translation of a higher-level rule into a standard-level rule is not possible according to the idea underlying $(\rightarrow E_{SL})$. However, this is not accidental: A reduction of higher-level to standard-level rules is not possible in general. A higher-level rule of the form

$$\begin{array}{c} [A \Rightarrow B] \\ \frac{C}{D} \end{array}$$

can be linearly written as $((A \Rightarrow B) \Rightarrow C) \Rightarrow D$. If it were, for fixed A, B, C, D, equivalent to a standard-level rule (with the same propositional letters), then $((A \Rightarrow B) \Rightarrow C) \Rightarrow D$ would be, viewed as a left-iterated implication $((A \rightarrow B) \rightarrow C) \rightarrow D$, equivalent in intuitionistic propositional logic to a conjunction-implication formula, in which implication is nested to the left maximally once. As for conjunction-implication logic the size of Kripke models to be considered for validity corresponds to the nesting of implications to the left, this is not the case. This does, of course, not preclude that in special cases, e.g., if certain letters are propositionally quantified (i.e., schematic), there is such an equivalence, such as, in the above example, between $\forall C : ((A \rightarrow B) \rightarrow C) \rightarrow C$ and $\forall C : (A \wedge (B \rightarrow C)) \rightarrow C$.

5 The benefit of higher-level rules: Strong uniformity and closure

In the following, we use the following terminology: 'generalized$_{\text{HL}}$' stands for 'generalized higher-level', generalized$_{\text{SL}}$' for 'generalized standard level'. When we speak of the higher-level' or the 'standard-level' approach, we always mean the approaches with *generalized* higher-level or standard-level rules, respectively, and similarly when we speak of 'higher-level' oder 'standard-level' natural deduction.

Both the higher-level and the standard-level generalized elimination rules satisfy the requirement that the elimination inferences for the standard connectives induced by the generalized form are equivalent to the 'common' elimination inferences, at least as inference schemata. In particular, modus ponens $(\rightarrow \mathrm{E_{MP}})$ is equivalent to the generalized forms $(\rightarrow \mathrm{E_{HL}})$ and $(\rightarrow \mathrm{E_{SL}})$. A second requirement one should impose on a generalized form is its uniformity: The elimination inferences covered by it should follow a uniform pattern — achieving this is the main purpose of the generalization. The generalized$_{\mathrm{HL}}$ rules obviously satisfy this criterion. The generalized$_{\mathrm{SL}}$ rules satisfy it to the extent that they give an 'indirect' reading to the elimination inferences expressed by the schematic minor premiss and conclusion C. However, the standard-level implication rule $(\rightarrow \mathrm{E_{SL}})$ is *hybrid* in so far as it has both the 'indirect' character expressed by the schematic C and the 'direct' character expressed by the minor premiss A which makes it a variant of modus ponens, if the derivation of C is trivialized to the derivation of B from B:

$$(\rightarrow \mathrm{E_{SL}})\ \frac{A \rightarrow B \quad A \quad [B]}{B}\ .$$

This hybrid character is the price one pays for avoiding higher levels, which would be inevitable otherwise. Therefore, the generalized$_{\mathrm{HL}}$ eliminations are *uniform in a stronger sense* than the standard-level rules. There is a third criterion which only the generalized$_{\mathrm{HL}}$ rules satisfy and which is a *closure property*: Unlike the standard-level approach, the higher-level approach allows us to formulate a general schema for elimination rules for *all* introduction rules that are possible on the basis of *all* available means of expression. In particular, the means of expression used to formulate a given elimination rule must be admitted to formulate a new introduction rule. Putting it negatively: An elimination rule must not be of a form that is not suitable in principle as an introduction rule. In other words, for every list of introduction rules that can be formulated at all, a corresponding elimination rule is available.

Of course, by extending the means of expression, for example by considering quantifiers or modal operators, one would be led to elimination rules of a different kind. However, every rule pattern used *in the given framework* should qualify to formulate introduction rules. That this is not the case with generalized$_{\mathrm{SL}}$ elimination rules in the manner of $(\rightarrow \mathrm{E_{SL}})$ is seen by the following example. Consider, for example, the rule

$$\frac{A \quad \begin{matrix}[B]\\ C\end{matrix}}{D}$$

which is a pattern used to formulate generalized$_{\mathrm{SL}}$ eliminations. Using this pattern as an introduction rule

$$(c_1\,\mathrm{I})\ \frac{A_1 \quad \begin{matrix}[A_2]\\ A_3\end{matrix}}{c_1(A_1, A_2, A_3)}$$

gives us a ternary operator c_1, for which there is a generalized$_{\mathrm{HL}}$ elimination rule

$$(c_1\,\mathrm{E})\quad \frac{c_1(A_1,A_2,A_3) \qquad \begin{matrix}[A_1 \quad A_2 \Rightarrow A_3]\\ C\end{matrix}}{C}\quad ,$$

but no generalized elimination rule according to the standard-level pattern. This situation might be remedied by using two separate generalized$_{\mathrm{SL}}$ elimination rules according to the standard-level pattern:

$$(c_1\,\mathrm{E})'\quad \frac{c_1(A_1,A_2,A_3) \qquad \begin{matrix}[A_1]\\ C\end{matrix}}{C} \qquad \frac{c_1(A_1,A_2,A_3) \qquad A_2 \qquad \begin{matrix}[A_3]\\ C\end{matrix}}{C}\quad .$$

However, this way out is not available if we consider the 4-ary connective c_2 with the introduction rules

$$(c_2\,\mathrm{I})\quad \frac{\begin{matrix}[A_1]\\ A_2\end{matrix}}{c_2(A_1,A_2,A_3,A_4)} \qquad \frac{\begin{matrix}[A_3]\\ A_4\end{matrix}}{c_2(A_1,A_2,A_3,A_4)}\quad .$$

The corresponding higher-level elimination rule is

$$(c_2\,\mathrm{E})\quad \frac{c_2(A_1,A_2,A_3,A_n) \qquad \begin{matrix}[A_1 \Rightarrow A_2]\\ C\end{matrix} \qquad \begin{matrix}[A_3 \Rightarrow A_4]\\ C\end{matrix}}{C}\quad .$$

This rule cannot be expressed along the lines of the standard-level rule $(\to\mathrm{E}_{\mathrm{SL}})$. The natural way of attempting such a solution would be to propose the following elimination rule:

$$(c_2\,\mathrm{E})'\quad \frac{c_2(A_1,A_2,A_3,A_4) \qquad A_1 \qquad \begin{matrix}[A_2]\\ C\end{matrix} \qquad A_3 \qquad \begin{matrix}[A_4]\\ C\end{matrix}}{C}\quad .$$

However, though it can be easily shown that $(c_2\,\mathrm{E})$ implies $(c_2\,\mathrm{E})'$ (the proof is similar to the demonstration that $(\to\mathrm{E}_{\mathrm{HL}})$ implies $(\to\mathrm{E}_{\mathrm{SL}})$ in section 3), the converse does not hold. Even if in a given application of $(c_2\,\mathrm{E})$ there is no vacuous discharge of $A_1 \Rightarrow A_2$ or $A_3 \Rightarrow A_4$ (in which case the minor premiss A_1 or A_3 of $(c_2\,\mathrm{E})'$ cannot be generated), it may happen that in the derivation of the minor premisses C of $(c_2\,\mathrm{E})$ an assumption *above* $A_1 \Rightarrow A_2$ or $A_3 \Rightarrow A_4$ is discharged at a rule application *below* $A_1 \Rightarrow A_2$ or $A_3 \Rightarrow A_4$, respectively, such as the B in the following example of a derivation of $(B \to A_2) \vee (B \to A_4)$ from $c_2(A_1,A_2,A_3,A_4)$, $B \to A_1$ and $B \to A_3$:

$$(c_2\,\mathrm{E})\,\frac{c_2(A_1,A_2,A_3,A_4) \qquad (\vee\,\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{{}^3[A_1 \Rightarrow A_2]\;\dfrac{(\to\mathrm{E}_{\mathrm{MP}})\,\dfrac{B \to A_1 \quad {}^1[B]}{A_1}}{A_2}}{B \to A_2}\,{}^1}{(B \to A_2)\vee(B \to A_4)} \qquad (\vee\,\mathrm{I})\,\dfrac{(\to\mathrm{I})\,\dfrac{{}^3[A_3 \Rightarrow A_4]\;\dfrac{(\to\mathrm{E}_{\mathrm{MP}})\,\dfrac{B \to A_3 \quad {}^2[B]}{A_3}}{A_4}}{B \to A_4}\,{}^2}{(B \to A_2)\vee(B \to A_4)}}{(B \to A_2)\vee(B \to A_4)}\,{}^3\quad .$$

Since $(c_2\,\mathrm{E})$ is a generalized$_{\mathrm{HL}}$ rule, whereas $(c_2\,\mathrm{E})'$ is considered as a generalized$_{\mathrm{SL}}$ rule, we have used applications of $(\to\mathrm{E_{MP}})$ (rather than applications of assumption rules), as they can easily be translated into either of these systems. The formula B occurs in both subderivations of the minor premisses in top position and thus *above* the assumptions $A_1 \Rightarrow A_2$ and $A_3 \Rightarrow A_4$, but is discharged at applications of $(\to\mathrm{I})$ *below* $A_1 \Rightarrow A_2$ and $A_3 \Rightarrow A_4$, so that we cannot split the subderivations of the minor premisses into two upper parts

$$(\to\mathrm{E_{MP}})\,\frac{B\to A_1 \quad B}{A_1} \qquad\qquad (\to\mathrm{E_{MP}})\,\frac{B\to A_3 \quad B}{A_3}$$

and two lower parts

$$(\vee\,\mathrm{I})\,\frac{(\to\mathrm{I})\,\dfrac{A_2}{B\to A_2}}{(B\to A_2)\vee(B\to A_4)} \qquad\qquad (\vee\,\mathrm{I})\,\frac{(\to\mathrm{I})\,\dfrac{A_4}{B\to A_4}}{(B\to A_2)\vee(B\to A_4)}$$

in order to obtain derivations of all four minor premisses of $(c_2\,\mathrm{E})'$.

This shows that the generalized$_{\mathrm{SL}}$ elimination rules do not cover the elimination rule necessary for such a simple connective as c_2, which has the meaning of $(A_1\to A_2)\vee(A_3\to A_4)$, although we can formulate its introduction rules in the standard-level framework. In this sense the generalized higher-level elimination rules are far more general than the standard-level ones.[7]

As indicated above, even higher-level rules do not suffice to capture every propositional connective. As an example we need not consider modal connectives and the like, but the ternary connective c_3 with the meaning $A_1\to(A_2\vee A_3)$ suffices. However the situation is entirely different from that of c_1 and c_2, as for c_3 there we cannot even give introduction rules of the form $(c_3\,\mathrm{I})$ using the means of expression available. (In this sense c_3 resembles, e.g., a modal operator.) As soon as we have introduction rules according to the schema $(c\,\mathrm{I})$, we do have a corresponding elimination rule $(c\,\mathrm{E})$. This is not the case with generalized$_{\mathrm{SL}}$ rules.[8]

[7] In his discussion of generalized left inferences in his higher-level sequent framework, Kutschera [11] (p. 15) gives an example similar to c_2 to show that the higher-level left rules cannot be expressed by lower-level left rules along the lines of the standard implication-left rule $(\to\mathrm{L})$ in the sequent calculus (which corresponds to $(\to\mathrm{E_{SL}})$, see below section 6).

[8] If we also consider operators *definable* in terms of others, i.e., if the premisses Δ_i of introduction rules $(c\,\mathrm{I})$ are allowed to contain operators c' which have already been given introduction and elimination rules, then c_3 is, of course, trivially definable, with

$$\frac{A_1\to(A_2\vee A_3)}{c_3(A1, A_2, A_3)}$$

as its introduction rule. This may be considered a rationale to confine oneself, as in the standard-level approach, to the standard operators. Such an approach fails, of course, to tell anything about the distinguished character of the standard operators as being capable to express all possible operators

For this greater uniformity we pay the price of introducing higher levels. Higher levels also lead to considerable simplifications in other areas, notably in type theory (see [5]).

6 Generalized elimination rules, sequent calculus and the proudness property

The motivations for using generalized$_{\mathrm{SL}}$ elimination rules vary between the authors mentioned in section 2. From the standpoint of proof-theoretic semantics, we [30, 33] proposed to use them in in a system called 'bidirectional natural deduction' that gives assumptions a proper standing. Another meaning-theoretical discussion of generalized$_{\mathrm{SL}}$ elimination rules can be found in [16]. In what follows we are only concerned with the formal relationship of generalized$_{\mathrm{SL}}$ elimination rules to left introduction rules in the sequent calculus along the lines advanced by von Plato and Tennant.

Generalized$_{\mathrm{SL}}$ elimination rules give natural deduction elimination rules a form which is familiar from left introductions in sequent calculi.[9] Whereas the right introduction rules directly correspond to the introduction rules in natural deduction, this is not so in the case of the usual eliminations rules. The rule of $\vee$ elimination can be read as corresponding to the left introduction rule of $\vee$ in the sequent calculus

$$\frac{A\vee B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C} \qquad\qquad \frac{\Gamma, A\vdash C \quad \Gamma, B\vdash C}{\Gamma, A\vee B\vdash C} \quad ,$$

but the standard $\wedge$ and $\rightarrow$ eliminations cannot. However, for the generalized$_{\mathrm{SL}}$ versions of these rules (for conjunction identical with the generalized$_{\mathrm{HL}}$ version), this is indeed the case:

$$\frac{A\wedge B \quad \begin{matrix}[A \quad B]\\ C\end{matrix}}{C} \qquad\qquad \frac{\Gamma, A, B\vdash C}{\Gamma, A\wedge B\vdash C}$$

$$\frac{A\rightarrow B \quad A \quad \begin{matrix}[B]\\ C\end{matrix}}{C} \qquad\qquad \frac{\Gamma\vdash A \quad \Gamma, B\vdash C}{\Gamma, A\rightarrow B\vdash C} \quad .$$

In this way a parallel between natural deduction and the sequent calculus is established.

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based on rules of a certain form. This is a point in which the goals of the generalized$_{\mathrm{HL}}$ and the generalized$_{\mathrm{SL}}$ approaches fundamentally differ from one another. (In [24, 26, 25] operators definable from other operators in the higher-level framework are considered, in addition to those definable without reference to others.)

[9] For the symmetry in Gentzen's sequent calculus and its description in terms of definitional reflection see [2, 34].

The parallel between the sequent calculus and natural deduction goes even further in the standard-level approach. It can be shown that every normal derivation based on generalized$_{\mathrm{SL}}$ elimination rules can be transformed into one, in which major premisses only occur in top position. To see this, we have to observe that every formula C, which is a conclusion of a (generalized$_{\mathrm{SL}}$) elimination inference and at the same time major premiss of a (generalized$_{\mathrm{SL}}$) elimination inference can be eliminated, as the following example demonstrates, which shows that corresponding segments[10] of formulas of this kind are shortened:

$$(\to \mathrm{E}_{\mathrm{SL}})\dfrac{(\to \mathrm{E}_{\mathrm{SL}})\dfrac{\begin{matrix}\mathcal{D}_1\\ D\to E\end{matrix}\quad\begin{matrix}\mathcal{D}_2\\ D\end{matrix}\quad\begin{matrix}{}^1[E]\\ \mathcal{D}_3\\ A\to B\end{matrix}}{A\to B}\,{}_1\quad\begin{matrix}\mathcal{D}_4\\ A\end{matrix}\quad\begin{matrix}{}^2[B]\\ \mathcal{D}_5\\ C\end{matrix}}{C}\,{}_2$$

reduces to

$$(\to \mathrm{E}_{\mathrm{SL}})\dfrac{\begin{matrix}\mathcal{D}_1\\ D\to E\end{matrix}\quad\begin{matrix}\mathcal{D}_2\\ D\end{matrix}\quad(\to \mathrm{E}_{\mathrm{SL}})\dfrac{\begin{matrix}{}^1[E]\\ \mathcal{D}_3\\ A\to B\end{matrix}\quad\begin{matrix}\mathcal{D}_4\\ A\end{matrix}\quad\begin{matrix}{}^2[B]\\ \mathcal{D}_5\\ C\end{matrix}}{C}\,{}_2}{C}\,{}_1 \quad .$$

Here it is assumed that below $A\to B$ there is no formula of the incriminated kind (in particular, C is not of that kind). Since maximal formulas (conclusions of introduction inferences being at the same time major premisses of (generalized$_{\mathrm{SL}}$) elimination inferences) are eliminated anyway, we can obtain a derivation of the required form.[11] This gives us a normal form theorem according to which every derivation based on generalized$_{\mathrm{SL}}$ elimination inferences can be transformed into a derivation with major premisses of elimination inferences standing always in top position. Following Tennant [35], who speaks of a major premiss of an elimination rules in top position, i.e., in assumption position, as *standing proud*, we call this property the *proudness property*:

PROUDNESS PROPERTY OF GENERALIZED$_{\mathrm{SL}}$ NATURAL DEDUCTION:
Every derivation in generalized$_{\mathrm{SL}}$ *natural deduction can be transformed into a derivation, in which every major premiss of an (generalized*$_{\mathrm{SL}}$*) elimination rule is an assumption.*

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[10]Following Prawitz [21], a segment is a succession of formula occurrences of the same form C such that immediately succeeding occurrences are minor premiss C and conclusion C of a generalized$_{\mathrm{SL}}$ elimination step.

[11]What one essentially does here, is carrying out permutative reductions as known from Prawitz [21]. Their general treatment, without assuming that sequents are maximal (and thus start with the conclusion of an introduction inference), but only that they end with an elimination inference, was proposed by Martin-Löf (see [22], p. 253f.) For the higher-level case, these reductions are used in [24].

A natural deduction derivation having the proudness property corresponds isomorphically (i.e., stepwise) to a cut-free derivation in the sequent calculus. Introduction of formulas in the sequent calculus on the left side of the sequent sign corresponds in generalized$_{\mathrm{SL}}$ natural deduction to introducing an assumption, which is the major premiss of an elimination rule.

The proudness property is not available without restriction in higher-level natural deduction. This is due to the fact that in the higher-level case we not only have *assumption formulas* but can also have *assumption rules*. Consider the following example of a derivation of B from $A \to B \wedge C$ and A:

$$(\to \mathrm{E}_{\mathrm{HL}})\ \frac{A \to B\wedge C \qquad {}^1[A \Rightarrow B\wedge C]\ (\wedge\,\mathrm{E}_{\mathrm{GEN}})\ \dfrac{\dfrac{A}{B\wedge C} \qquad {}^2[B]}{B}\,{}_2}{B}\,{}_1 \quad . \tag{8}$$

Another example is the following derivation of C from $A \to (B \to C)$, A and C:

$$(\to \mathrm{E}_{\mathrm{HL}})\ \frac{A \to (B \to C) \qquad {}^1[A \Rightarrow (B \to C)]\ (\to \mathrm{E}_{\mathrm{HL}})\ \dfrac{\dfrac{A}{B \to C} \qquad {}^2[B \Rightarrow C]\ \dfrac{B}{C}}{C}\,{}_2}{C}\,{}_1 \quad . \tag{9}$$

In (8) the formula $B\wedge C$ is the conclusion of an assumption rule and at the same time the major premiss of an elimination rule, a situation which cannot be further reduced. Similarly, in (9) the formula $B \to C$ is the conclusion of an assumption rule and at the same time the major premiss of an elimination rule. However, if we weaken the notion of proudness to include this situation, then derivations in the higher-level approach satisfy it. We call a formula occurrence, which is a conclusion of the application of an assumption rule and at the same time major premiss of an elimination rule *weakly proud*. Then the following *weak proudness property* of higher-level natural deduction holds:

Weak proudness property of generalized$_{\mathrm{HL}}$ natural deduction:
Every derivation in generalized$_{\mathrm{HL}}$ natural deduction can be transformed into a derivation, in which every major premiss of an elimination rule is either an assumption or a conclusion of an assumption rule[12].

For example, the following derivation, in which $B\wedge C$ occurs both as a conclusion of an elimination inference and as the major premiss of another elimination inference, can be reduced to (8):

$$(\wedge\,\mathrm{E}_{\mathrm{GEN}})\ \frac{(\to \mathrm{E}_{\mathrm{HL}})\ \dfrac{A \to B\wedge C \qquad {}^1[A \Rightarrow B\wedge C]\ \dfrac{A}{B\wedge C}}{B\wedge C}\,{}_1 \qquad {}^2[B]}{\mathrm{B}}\,{}_2 \quad .$$

[12]As an assumption formula can be viewed as the conclusion of a first-level assumption rule, the second alternative actually includes the first one.

Similarly, the following derivation, in which $B \to C$ occurs both as a conclusion of an elimination inference and as the major premiss of another elimination inference, can be reduced to (9):

$$(\to \mathrm{E_{HL}})\ \dfrac{(\to \mathrm{E_{HL}})\ \dfrac{A \to (B \to C) \qquad {}^{1}[A \Rightarrow (B \to C)]\ \dfrac{A}{B \to C}}{B \to C}\ {}_{1} \qquad {}^{2}[B \Rightarrow C]\ \dfrac{\mathrm{B}}{C}}{\mathrm{C}}\ {}_{2} \quad .$$

Weak proudness implies the subformula property. This is due to the fact that every formula C standing weakly proud is the subformula of an assumption rule. If this assumption rule is undischarged, then C is a subformula of an open assumption anyway. Otherwise, this assumption rule is discharged at a higher-level introduction or elimination rule, in which case it is a subformula of the formula introduced or eliminated.[13] If one considers the subformula principle to be the fundamental corollary of normalization, the fact that for higher-level natural deduction we only have the weak but not the full proudness property, is no real disadvantage as compared to the standard-level (generalized) alternative.

7 A sequent calculus variant of the higher-level approach

Although intended for the purpose of uniform elimination rules in natural deduction, one might investigate how the generalized$_{\mathrm{HL}}$ approach fits into a sequent-style framework and which form the weak proudness property then takes, in comparison to the proudness property of the standard-level approach, which immediately corresponds to cut-free derivations. Some of the ideas and results presented here have been established by Avron [1], others can be found in [27] and [7].[14]

We only consider the single-succedent (intuitionistic) variant. The generalized conjunction rule ($\wedge\, \mathrm{E_{GEN}}$) simply translates into the $\wedge$ left rule

$$(\wedge\, \mathrm{L})\ \dfrac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ ,$$

the only difference being that Γ may now stand for a set of higher-level rules rather than only for a set of formulas.[15] In order to frame the idea of assumption rules, we

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[13]For a detailed proof of normalization and subformula property for higher-level natural deduction see [24].

[14]Kutschera's [11] approach using an iteration of the sequent arrow should be mentioned as well, although it needs some reconstruction to fit into our framework.

[15]We can here neglect the difference between lists, multisets and sets, as for simplicity we always assume that the usual structural rules of the intuitionistic sequent calculus (permutation, contraction and thinning) are available — with the exception of cut, whose availability or non-availability as a primitive or admissible rule will always be explicitly stated.

have to introduce a schema for the left introduction of a rule R as an assumption. We assume that R has the following general form:

$$(\Delta_1 \Rightarrow A_1, \ \ldots \ , \Delta_n \Rightarrow A_n) \ \Rightarrow \ B \ ,$$

which covers the limiting case $n = 0$ in which R is just the formula (= level-0-rule) B, and the cases in which some or all Δ_i, which are lists of rules, are empty, i.e., in which $\Delta_i \Rightarrow A_i$ is the same as A_i. Then the left introduction of a rule, which corresponds to using a rule as an assumption in generalized$_{\mathrm{HL}}$ natural deduction (1), reads as follows:

$$(\Rightarrow \mathrm{L}) \ \frac{\Gamma, \Delta_1 \vdash A_1 \quad \ldots \quad \Gamma, \Delta_n \vdash A_n}{\Gamma, ((\Delta_1 \Rightarrow A_1, \ \ldots \ , \Delta_n \Rightarrow A_n) \ \Rightarrow \ B) \vdash B} \ ,$$

which covers as a limiting case:

$$(\Rightarrow \mathrm{L})^\circ \ \frac{\Gamma \vdash A}{\Gamma, (A \Rightarrow B) \vdash B} \ .$$

The right and left rules for implication are then the following:

$$(\rightarrow \mathrm{R}) \ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \qquad (\rightarrow \mathrm{L}_{\mathrm{HL}}) \ \frac{\Gamma, A \Rightarrow B \vdash C}{\Gamma, A \rightarrow B \vdash C} \ .$$

The sequent calculus with higher-level rules, which results from the ordinary sequent calculus with cut by adding $(\Rightarrow \mathrm{L})$, and by using $(\rightarrow \mathrm{R})$ and $(\rightarrow \mathrm{L}_{\mathrm{HL}})$ as rules for implication, will be called the *higher-level sequent calculus*. So we disregard here the feature that in addition to $\rightarrow$, $\wedge$ and $\vee$ we have right and left introduction rules for n-ary connectives and consider just the standard connectives, as this is the concern of the standard-level approach.

Obviously, every derivation in the higher-level sequent calculus (with cut) can be translated into higher-level natural deduction, as the left-introduction rules are available as generalized$_{\mathrm{HL}}$ inferences, and $(\Rightarrow \mathrm{L})$ is available as the introduction of an assumption rule. Conversely, every derivation in higher-level natural deduction can be translated into the higher-level sequent calculus (with cut) along the lines described by Gentzen ([6], p. 422–424): Applications of introduction rules, of assumption rules, and of elimination rules with major premisses standing proud are homophonically translated into applications of right introduction rules, of $(\Rightarrow \mathrm{L})$, and of left introduction rules, respectively. Only in the situation in which the major premiss of an elimination inference is not standing proud:

$$\text{E inference} \ \frac{\begin{matrix}\mathcal{D} \\ A\end{matrix} \quad \begin{matrix}\mathcal{D}_1 \\ C\end{matrix} \ \ldots \ \begin{matrix}\mathcal{D}_n \\ C\end{matrix}}{C} \ ,$$

we must apply cut, yielding

$$\text{Cut} \ \frac{\begin{matrix}\mathcal{D}' \\ \Gamma \vdash A\end{matrix} \qquad \text{L inference} \ \dfrac{\mathcal{D}'_1 \ \ldots \ \mathcal{D}'_m}{\Delta, A \vdash C}}{\Gamma, \Delta \vdash C} \ ,$$

where $\mathcal{D}', \mathcal{D}'_1, \ldots, \mathcal{D}'_m$ are the sequent calculus translations of $\mathcal{D}, \mathcal{D}_1, \ldots, \mathcal{D}_m$. (This procedure works for arbitrary n-ary connectives, too.)

The weak proudness property of higher-level natural deduction gives us immediately a *weak cut elimination theorem*:

WEAK CUT ELIMINATION FOR THE HIGHER-LEVEL SEQUENT CALCULUS: *Every derivation in the higher level sequent calculus (with cut) can be transformed into a derivation, in which cut occurs only in the situation, where its left premiss is the conclusion of an application of an assumption rule, and the right premiss the conclusion of a left introduction rule for the cut formula, i.e. only in the following situation*

$$(\text{Cut})\ \dfrac{(\Rightarrow\text{L})\ \dfrac{\vdots}{\Gamma \vdash A} \qquad (\text{L inference for } A)\ \dfrac{\vdots}{\Delta, A \vdash C}}{\Gamma, \Delta \vdash C}\ .$$

As we have the subformula principle for higher-level natural deduction, it holds for the higher-level sequent calculus as well, if we only allow for cuts of the form described in the weak cut elimination theorem. Therefore cuts of this special form are harmless, although perhaps not most elegant.

That we do not have full cut elimination is demonstrated by the sequent-calculus translation of our example (8):

$$(\to \text{L}_{\text{HL}})\ \dfrac{(\text{Cut})\ \dfrac{(\Rightarrow\text{L})\ \dfrac{A \vdash A}{A, (A \Rightarrow B \wedge C) \vdash B \wedge C} \qquad (\wedge\text{L})\ \dfrac{B \vdash B}{B \wedge C \vdash B}}{A, (A \Rightarrow B \wedge C) \vdash B}}{A, (A \to B \wedge C) \vdash B}\ . \tag{10}$$

As there is no inference rule (apart from cut) which can generate $A, (A \Rightarrow B \wedge C) \vdash B$ (at least if A, B and C are atomic and different from one another), cut is not eliminable. However, this application of cut is of the form permitted by the weak cut elimination theorem.

It might appear asymmetric at first glance that there is a left introduction rule ($\Rightarrow$L), but no right introduction rule for the rule arrow $\Rightarrow$. Why do we not have a right introduction rule for $\Rightarrow$ of the form

$$(\Rightarrow\text{R})\ \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}$$

and then introduce implication $\to$ on the right directly in terms of $\Rightarrow$, as it is done on the left:

$$(\to\text{R})'\ \dfrac{\Gamma \vdash A \Rightarrow B}{\Gamma \vdash A \to B} \quad ?$$

This asymmetry: only a left rule for $\Rightarrow$, is due to the fact that the rule arrow is not a logical constant in the genuine sense but a sign belonging to the structural apparatus, comparable to the comma. If we look at the comma and the rules for conjunction, we observe a similar phenomenon:

$$(\wedge\,\mathrm{R})\,\frac{\Gamma\vdash A\quad \Gamma\vdash B}{\Gamma\vdash A\wedge B}\qquad (\wedge\,\mathrm{L})\,\frac{\Gamma, A, B\vdash C}{\Gamma, A\wedge B\vdash C}\ .$$

When applying $(\wedge\,\mathrm{R})$, we do not first introduce a comma on the right hand side, which is conjunctively understood, in a way like

$$\frac{\Gamma\vdash A\quad \Gamma\vdash B}{\Gamma\vdash A, B}\ ,$$

and then introduce $\wedge$ in terms of the comma in a manner such as

$$\frac{\Gamma\vdash A, B}{\Gamma\vdash A\wedge B}\ .$$

Rather, we have a direct right-introduction rule $(\wedge\,\mathrm{R})$ for conjunction, whereas on the left side, by means of $(\wedge\,\mathrm{L})$, conjunction is reduced to the comma. This asymmetry is somewhat concealed by the fact that there is no formal left introduction rule for the comma in the sense in which there is a formal left introduction rule for the rule arrow $\Rightarrow$, as the comma is already there as a means to separate antecedent formulas (or antecedent rules). It nevertheless is a structural entity governed by rules which do not fit into the symmetric right-left-introduction schema. Analogously, the rule arrow must be looked upon as an enrichment of structural reasoning which essentially affects only the left side of the turnstile (in the intuitionistic framework).

The idea of higher-level rules, i.e. of rules as assumptions, is that we enrich our possibilities of formulating assumptions, in order to characterize logical operators as having the same consequences as certain assumption structures. Here $A\wedge B$ has the same consequences as the assumption structure (A, B), and $A\rightarrow B$ has the same consequences as the assumption structure $A\Rightarrow B$.

Though against the spirit of using rules as assumptions (or members of the antecedent), it is possible to express rules in terms of the standard operators. In fact, such a translation is used if we want to show that the standard operators suffice to express everything that can be expressed by means of higher-level rules, i.e., if we establish the expressional completeness[16] of the standard operators (see [11, 24, 26, 25, 27]). We translate the rule arrow $\Rightarrow$ by means of implication $\rightarrow$ and the comma by conjunction $\wedge$, so that, for example, a rule

$$(A, B\Rightarrow C), (E\Rightarrow F)\ \Rightarrow\ G$$

becomes the implication

$$(A\wedge B\rightarrow C)\wedge(E\rightarrow F)\ \rightarrow\ G\ .$$

[16]This corresponds to functional completeness in the case of truth functions.

If we use this translation, then ($\rightarrow \mathrm{L_{HL}}$) becomes superfluous as premiss and conclusion are identical, and rule ($\Rightarrow$ L) becomes

$$\frac{\Gamma, \Delta_1 \vdash A_1, \ \ldots \ \Gamma, \Delta_n \vdash A_n}{\Gamma, ((\Delta_1 \rightarrow A_1) \wedge \ \ldots \ \wedge (\Delta_n \rightarrow A_n) \ \rightarrow \ B) \vdash B} \ . \tag{11}$$

This rule can be replaced by the simpler rule

$$(\rightarrow \mathrm{L})^\circ \ \frac{\Gamma \vdash A}{\Gamma, (A \rightarrow B) \vdash B}$$

which corresponds to ($\Rightarrow$ L)$^\circ$. Given the premiss of (11), we simply need to use ($\rightarrow$ R) and ($\wedge$ R) to obtain

$$\Gamma \vdash (\Delta_1 \rightarrow A_1) \wedge \ \ldots \ \wedge (\Delta_n \rightarrow A_n)$$

from which by means of ($\rightarrow$ L)$^\circ$ we obtain the conclusion of (11).

The result is a sequent calculus, in which the common ($\rightarrow$ L) rule

$$(\rightarrow \mathrm{L}) \ \frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma, A \rightarrow B \vdash C}$$

is replaced with ($\rightarrow$ L)$^\circ$. ($\rightarrow$ L)$^\circ$ introduces the spirit of *rules* into a sequent calculus of the usual kind (i.e., without higher-level rules): From A we may pass over to B by using (= assuming) $A \rightarrow B$ understood as a rule which licenses this transition.

It is obvious that ($\rightarrow$ L)$^\circ$ and ($\rightarrow$ L) are interderivable, if the letter C is understood schematically, i.e., can be replaced with B. We call the sequent calculus with ($\rightarrow$ L)$^\circ$ as the left introduction rule for implication the *sequent calculus based on the implications-as-rules interpretation*, in short *rule-style sequent calculus* as opposed to the standard sequent calculus which has ($\rightarrow$ L) as left introduction rule. As it results by translation from the higher-level sequent calculus, we do not have cut elimination for this system. As a translation of (10), the following is a counterexample:

$$\frac{(\rightarrow \mathrm{L})^\circ \ \dfrac{A \vdash A}{A, (A \rightarrow B \wedge C) \vdash B \wedge C} \qquad (\wedge \mathrm{L}) \ \dfrac{B \vdash B}{B \wedge C \vdash B}}{A, (A \rightarrow B \wedge C) \vdash B} \ (\mathrm{Cut}) \ . \tag{12}$$

However, corresponding to the weak proudness property of higher-level natural deduction and the weak cut elimination theorem in the higher-level sequent calculus, we have a weak cut elimination theorem for the rule-style sequent calculus, which says that a situation such as (12) is essentially the only one where cuts must be admitted.

Weak cut elimination for the rule-style sequent calculus:
Every derivation in the rule-style sequent calculus (with cut) can be transformed into a derivation, in which cut occurs only in the situation, where its left premiss is the

conclusion of $(\rightarrow \mathrm{L})^\circ$*, and where its right premiss results from introducing the cut formula in the last step, i.e., in the following situation:*

$$\dfrac{(\rightarrow \mathrm{L})^\circ \dfrac{\vdots}{\Gamma \vdash A} \qquad (\text{L inference for } A) \dfrac{\vdots}{\Delta, A \vdash C}}{\Gamma, \Delta \vdash C}\,(\mathrm{Cut}) \quad . \tag{13}$$

In fact, if we consider a purely implicational system with $(\rightarrow \mathrm{L})^\circ$ of the multi-ary form

$$\dfrac{\Gamma \vdash A_1 \quad \ldots \quad \Gamma \vdash A_n}{\Gamma, A_1 \rightarrow (\ldots (A_n \rightarrow B)\ldots) \vdash B}$$

then we have full cut elimination, as remarked by Avron [1].[17] Analogously, the purely implicational natural deduction system with the following rule for implication

$$\dfrac{A_1 \rightarrow (\ldots (A_n \rightarrow B)\ldots) \rightarrow B \quad A_1 \quad \ldots \quad A_n}{B}$$

enjoys the full proudness property.

The rule-style sequent calculus satisfies the *subformula principle*, in spite of the weak form of cut which cannot be eliminated. This result is also carried over from the consideration of explicit higher-level rules. It is immediately plausible, too, as the cut formula A in (13) is contained in an implication $B \rightarrow A$ which is introduced by means of $(\rightarrow \mathrm{L})^\circ$ and therefore belongs to Γ.

8 *Implications-as-rules* vs. *implications-as-links*

We have seen that, formalized as a sequent calculus, the interpretation of implications as rules yields a system with $(\rightarrow \mathrm{L})^\circ$ as the left introduction rule for implication. This system enjoys the subformula property, but only a weak form of cut elimination. Although the implications-as-rules view is very natural in the natural deduction framework, the corresponding rule-style sequent calculus might look strange at first glance, as one has become used to Gentzen's rule $(\rightarrow \mathrm{L})$ and to full cut elimination as a fundamental principle.

However, the alleged simplicity of $(\rightarrow \mathrm{L})$ is essentially a feature of *technical elegance*. If we want to have full cut elimination at any price in order to derive its corollaries such as the subformula property and other features with ease, then $(\rightarrow \mathrm{L})$ is the rule of

[17] Avron also remarks that the standard $(\rightarrow \mathrm{L})$ rule is a way of avoiding the multi-ary character of this rule, which cannot be effected by means of $(\rightarrow \mathrm{L})^\circ$ alone (if conjunction is not available). Negri and von Plato [17] (p. 184) mention the rule $(\rightarrow \mathrm{L})^\circ$ as a sequent calculus rule corresponding to modus ponens, followed by a counterexample to cut analogous to (12), which is based on implication only. This counterexample shows again that for cut elimination in the implicational system the multi-ary form of $(\rightarrow \mathrm{L})^\circ$ considered in [1] and the corresponding forms of rule introduction in the antecedent considered in [27] and [7] are really needed.

choice. In fact, it were these technical considerations that led Gentzen to consider his sequent calculus. Unlike the calculus of natural deduction, for which Gentzen coined the term 'natural', and for whose rules he gave a detailed philosophical motivation, for the sequent calculus he did not claim philosophical plausibility, but merely its being suited for the proof of the Hauptsatz (see [6], p. 191).

If we look at Gentzen's rule ($\rightarrow$L) from a philosophical or conceptual point of view and compare it to the implications-as-rules view, then it looses some of its plausibility. Whereas the implications-as-rules interpretation gives a direct meaning to implication, as the notion of a rule is a very basic notion used to describe reasoning, and acts in general, from an elementary perspective, this does not hold for Gentzen's notion of implication as formalized in the sequent calculus. One can even argue that the feature of full cut elimination, which is distinctive of Gentzen's sequent calculus, is enforced by or at least embodied in his particular formulation of ($\rightarrow$L), in contradistinction to ($\rightarrow$L)°. Translated into natural deduction, an application ($\rightarrow$L)° can be displayed as follows:

$$A \rightarrow B \, \frac{\begin{array}{c}\mathcal{D}_1\\ A\end{array}}{B} \quad . \tag{14}$$

As it represents the implications-as-rules interpretation, we write $A \rightarrow B$ as labelling the transition from A to B. The interpretation of implication underlying ($\rightarrow$L) would then be displayed as

$$A \rightarrow B \, \frac{\begin{array}{c}\mathcal{D}_1\\ A\end{array}}{\begin{array}{c}B\\ \mathcal{D}_2\\ C\end{array}} \quad . \tag{15}$$

Whereas in the first case, $A \rightarrow B$ just licenses to continue from A to B by extending $\mathcal{D}_1$, in the second case it *links* two derivations, namely the derivation $\mathcal{D}_1$ of A and the derivation $\mathcal{D}_2$ from B. In this sense we can speak of the *implications-as-links* interpretation as opposed to the *implications-as-rules* interpretation. Obviously, the implications-as-links interpretation adds to the implications-as-rules interpretation something that is expressed by the rule of cut, viz. the cut with cut formula B. Passing from (14) to (15) requires an implicit step which is expressed by the cut rule. So the fact that the implications-as-links interpretation leads to full cut elimination, is due to the fact that it embodies already some limited form of cut which in the implications-as-rules interpretation would have to be added separately.

That an implications-as-links view underlies Gentzen's sequent calculus is supported by the fact that even in systems, in which cut elimination fails, for example in systems with additional axioms or rules, a cut with the formula A can be enforced by adjoining

$A \to A$ to the antecedent, as cut then then becomes an instant of $(\to \mathrm{L})$:

$$(\to \mathrm{L})\ \ \frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma, A \to A \vdash B}\ .$$

From the standpoint of the implications-as-rules interpretation this effect is quite unplausible, as assuming the rule that allows one to pass over from A to A should be vacuous and have no effect whatsoever. The fact that adding $A \to A$ immediately enables cut with A shows that implication has received an interpretation which relates it to cut.

This argument not only shows that the implications-as-rules interpretation is more plausible than the implications-as-links interpretation, but also more elementary. As the implications-as-links interpretation adds certain features of cut to the implications-as-rules interpretation, it is, from the philosophical point of view, strongly advisable to separate these two features, i.e., to use $(\to \mathrm{L})^{\circ}$ as the more elementary rule for implication, together with cut in a weakened form.

We should emphasize that these results apply to the intuitionistic case only. As soon as we consider the linear or classical case with more than one formula permitted in the succedent, different considerations apply which do not necessarily favour the implications-as-rules view, but might speak for the implications-as links view, due to the symmetry inherent in the multiple-succedent sequent calculus (see [34]). However, in such systems implication has a different meaning, and, from a conceptual point, it is questionable if there is a genuine notion of implication at all.

Concluding, the implications-as-links interpretation has substantial support from the simplicity of the underlying sequent calculus and its cut elimination feature, so for many technical considerations the implications-as-links interpretation is preferable. However, the implications-as-rules interpretation has the conceptual merit of carrying over the naturalness of natural deduction and the naturalness of the concept of a rule to the sequent calculus, including its natural extension by higher-level rules which allows for a very general treatment of logical constants. It motivates a sequent calculus which is not cut free, but needs a weak version of cut which does not obstruct the subformula principle .

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