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# Definitional Reasoning in Proof-Theoretic Semantics and the Square of Opposition

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**Abstract**

Within a framework of reasoning with respect to clausal definitions of atoms, four forms of judgement are distinguished: Direct and indirect assertion, and direct and indirect denial. Whereas direct assertion and direct denial are established by directly applying a definitional clause ("definitional closure"), indirect assertion and indirect denial result from showing that all possible premisses of the opposite judgement can be refuted ("definitional reflection"). The deductive relationships between these four forms of judgement correspond to those represented in the square of opposition, if direct assertion and direct denial are placed in the $A$ and $E$ corners, and indirect assertion and indirect denial in the $I$ and $O$ corners of the square.

## 1. Introduction

The square of opposition can be looked at from various points of view. This paper will only be concerned with the deductive relationships between its nodes. In our inferential framework we shall define four forms of judgement to be placed in the $A, E, I$ and $O$ corners of the square of oppositon, as well as a denial operation between judgements, such that

- the judgements in the $A$ and $E$ corners oppose each other without necessarily being denials of one another — *contrariety*,

- the judgements in the $A$ and $O$ corners, and those in the $E$ and $I$ corners are denials of one another — *contradictoriness*,
- the judgements in the $I$ and $O$ corners can be inferred from those in the $A$ and $E$ corners, respectively, but not necessarily vice versa — *subalternation*,
- the denials of the judgements in the $I$ and $O$ corners oppose each other, without necessarily being denials of one another — *subcontrariety*.

*Proof-theoretic semantics* denotes an approach to reasoning according to which the semantics of propositions is given in terms of proofs, and in which proofs themselves are philosophically primary. This may be viewed as an interpretation of the slogan "meaning is use", if proofs are related to the use of linguistic expressions. There are several variants of proof-theoretic semantics. The "standard" one, which is due to Prawitz and Dummett, is based on a proof-theoretic notion of validity and is described in detail in [11]. Here we are concerned with a somewhat different, and more general, approach developed jointly with Lars Hallnäs, originally in the context of logic programming [7, 9]. It is called "definitional reasoning" and investigates rule-based reasoning in a broad sense, not just logical reasoning, i.e. reasoning with logical constants.

## 2. Clausal definitions

Unlike other approaches, which deal with logically complex formulas, ours deals with atomic formulas or *atoms*, for which a clausal definition is given. For the sake of simplicity, and in order to make the basic points as clear as possible, we only consider the very simple propositional case without any term structure and without individual variables. Therefore atoms are the same as propositional veriables, which will be denoted by lower case letters. Upper case letters denote (possibly empty) lists of atoms. A clausal definition, in short *definition*, has the form

$$\mathbb{D}\left\{\begin{array}{lcl} a_1 & \Leftarrow & B_{11} \\ & \vdots & \\ a_1 & \Leftarrow & B_{1m_1} \\ & \vdots & \\ a_n & \Leftarrow & B_{n1} \\ & \vdots & \\ a_n & \Leftarrow & B_{nm_n} \end{array}\right. .$$

As usual in logic programming, the heads of the clauses are put in front, and clauses are read as:

$$\begin{array}{lll} a_1 & \textit{if} & B_{11} \\ a_1 & \textit{if} & B_{12} \\ & \vdots & \end{array} .$$

The reader who is not familiar with this notation, may use the left-to-right notation

$$\mathbb{D} \left\{ \begin{array}{lcl} B_{11} & \Rightarrow & a_1 \\ & \vdots & \\ B_{1m_1} & \Rightarrow & a_1 \\ & \vdots & \\ B_{n1} & \Rightarrow & a_n \\ & \vdots & \\ B_{nm_n} & \Rightarrow & a_n \end{array} \right.$$

to be read as

$$\mathbb{D} \left\{ \begin{array}{llll} \mathit{if} & B_{11} & \mathit{then} & a_1 \\ \mathit{if} & B_{12} & \mathit{then} & a_1 \\ & & \vdots & \end{array} \right. .$$

However, it is important to be aware that these clauses are not understood as (material) implications, but rather as production rules. So

$$a \ \Leftarrow \ B$$

is intended to mean that from $B$ one may produce $a$. In order to avoid any confusion with inference rules, which later on will be defined *with respect to* such entities, but otherwise are different from them, we avoid the term "rule" and speak of "clauses" instead, again following the terminology of logic programming. However, here another *caveat* must be put in place: We do not mean "clause" in the particular sense of resolution theory where it is often understood as a disjunction of literals. Clauses in our sense do not presuppose logic. Understanding them as elementary productions, i.e. devices that generate their heads from their bodies, is sufficient. The "real" meaning of clauses is explained by the inference rules motivated in the next two sections.

Bodies of clauses are also called *conditions*. If a clause has the form

$$a \ \Leftarrow \ B$$

then $B$ is called a condition of $a$. As bodies may be empty, a clause has either the form

$$a \ \Leftarrow \ b_1, \ldots, b_n$$

or the form

$$a \ \Leftarrow$$

As there may be more than one clause for $a$, $a$ may have more than one condition according to a given definition. In this sense definitions are not necessarily deterministic. For example

$$\begin{cases} a & \Leftarrow & b, c \\ a & \Leftarrow & d \\ a & \Leftarrow & e, f, g, h \end{cases}$$

is a definition of $a$ in terms of three conditions, using three clauses for $a$. Thus definitions are not to be confounded with explicit definitions in the usual sense, which can be read bidirectionally. Definitions in our sense correspond to *inductive definitions* in the mathematical sense. In fact, in a general setting, inductive definitions are best understood as systems of clauses in our sense, by means of which a domain of objects can be generated (see [1]). Clauses themselves are logic-free, except that they are based on some *structural* logic in the sense that the comma within conditions is understood conjunctively.

## 3. Putting definitions into action: definitional closure and definitional reflection

The way clauses are understood is determined by *inference rules* which handle them, put them into action or make their meaning explicit (to use three different phrases for what is intended). The simplest inference rule says that if we can assert the defining conditions of $a$, then we can assert $a$ itself. If

$$a \;\Leftarrow\; b_1, \ldots, b_n$$

is a defining clause for $a$, this clause can be applied using the inference rule

$$\frac{b_1 \;\ldots\; b_n}{a}\,,$$

which, as a limiting case, covers

$$\frac{}{a}\;.$$

This rule is an introduction rule for $a$, as it allows us to introduce $a$ from one of its conditions. As a general schema it is called the rule of *definitional closure*, as it enables us to close a set of given atoms under the clauses of a definition. It allows us to build proof trees from given definitions. For example, the definition

$$\mathbb{D}_1 \begin{cases} a & \Leftarrow & b, c \\ b & \Leftarrow & \\ c & \Leftarrow & \\ d & \Leftarrow & a, b \\ e & \Leftarrow & a, f, h \\ h & \Leftarrow & g \end{cases}$$

allows us to derive $a$ in $\mathbb{D}_1$ as follows:

$$\dfrac{\dfrac{\dfrac{}{b} \quad \dfrac{}{c}}{a} \quad \dfrac{}{b}}{d} \quad .$$

As usual we write

$$\vdash_{\mathbb{D}_1} d$$

to express the corresponding provability statement. An example of a proof of an atom from assumptions would be the following

$$\dfrac{\dfrac{\dfrac{}{b} \quad \dfrac{}{c}}{a} \quad f \quad \dfrac{g}{h}}{e}$$

establishing

$$f, g \vdash_{\mathbb{D}_1} e.$$

Graphically, an assumption occurs as an atom without an inference bar on top of it.

This idea of reasoning with definitions can be extended by also giving elimination rules for the atoms defined. Whereas the introduction rules are very elementary — they just interpret clauses as productions —, generalizing the idea of elimination rules from logic to clausal systems makes definitional reasoning particularly powerful. This idea was called "definitional reflection" by Hallnäs and is described in [5, 6, 7, 9, 10, 12]. Here we discuss instead the idea of *denial rules* which is based on definitional reflection as well.

Suppose with every atom $a$ there is associated its denial $-a$ in a sense that makes $-a$ the contradictory opposite to $a$. More precisely, suppose there is a metalinguistic denial operation $\widehat{\ }$ such that

$$\begin{array}{lll} \widehat{a} & := & -a \\ \widehat{-a} & := & a \quad . \end{array}$$

Thus we assume that with each atom $a$ there is associated its (objectlinguistic) denial $-a$, such that $a$ and $-a$ are metalinguistic denials of one another. To avoid the (object- vs. meta-linguistic) ambiguity in the term "denial", we sometimes speak of the *complement* $\widehat{\ell}$ of a *literal* $\ell$, where literals are atoms $a$ and their denials $-a$. As our framework is quite general, we do not presuppose any particular understanding of "denial", as long as a complementation operation with the property mentioned is available. One way of understanding it (but not the only, and definitely not the most central one) is in the sense of classical Boolean negation. Note that in the simplified framework dealt with here, we do not draw a distinction between a pragmatic denial operation (which cannot be iterated) and the negation connective (which may occur nested inside a formula), as the only formulae we are dealing with here are literals, i.e. atoms and their denials. We are interested in the way denials of atoms gives rise to new inference rules without changing the clausal definition (in which, for the time being, no denial is allowed to occur).

Our motivation is that, if the atom $a$ is defined by the clauses

$$\begin{cases} a & \Leftarrow & B_1 \\ & \vdots & \\ a & \Leftarrow & B_n \end{cases}$$

then, if asserting *one* of $a$'s defining conditions suffices to assert $a$ ($a$ *introduction*), denying *all* defining conditions of $a$ suffices to deny $a$, i.e. to establish $-a$ ($-a$ introduction). This leads to a rule of the form

$$\frac{\widehat{B_1} \quad \ldots \quad \widehat{B_n}}{-a}\ (-) \quad .$$

For this rule expression to make sense, we have to define the denial (or complement) $\widehat{B}$ of a condition $B$. As a condition is a list $b_1, \ldots, b_n$ of atoms, it is natural to consider it to be rejected, if one of its elements is rejected. So we let $\widehat{b_1, \ldots, b_n}$ stand indefinitely for the denial $\widehat{b_i}$ of anyone of its members, so that every proof of $-b_i$ for some element $b_i$ counts as a proof of $\widehat{b_1, \ldots, b_n}$. Since we do not want to introduce $\widehat{b_1, \ldots, b_n}$ as a formal expression into our object-language, we adopt the following convention: When we state

$$\frac{\widehat{B}}{a}$$

as an inference rule, where $B$ is $b_1, \ldots, b_n$, then this stands for the list of inference rules

$$\frac{-b_1}{a} \quad \ldots \quad \frac{-b_n}{a} \quad ,$$

and the notation for the schema of $-a$ introduction

$$\frac{\widehat{B_1} \quad \ldots \quad \widehat{B_n}}{-a}\ (-)$$

stands for the list of all possible rules of the form

$$\frac{-b'_1 \quad \ldots \quad -b'_n}{-a}$$

where $b'_i$ is arbitrarily selected from $B_i$ for every $i$ ($1 \leq i \leq n$). Related expressions occurring later in this paper are understood accordingly. For example, if $a$ is defined by the clauses

$$\begin{cases} a & \Leftarrow & b, c \\ a & \Leftarrow & d \\ a & \Leftarrow & e, f \end{cases}$$

then the schema of $-a$ introduction $(-)$ stands for the following inference rules:

$$\frac{-b \quad -d \quad -e}{-a} \quad \frac{-b \quad -d \quad -f}{-a} \quad \frac{-c \quad -d \quad -e}{-a} \quad \frac{-c \quad -d \quad -f}{-a} \quad .$$

In the formulation of $(-)$ it is important that $B_1, \ldots, B_n$ *exhaust* the conditions of $a$. If there were a further condition $B_{n+1}$ of $a$, based on the additional clause

$$a \Leftarrow B_{n+1}$$

we would have to extend this rule to

$$\frac{\widehat{B_1} \quad \ldots \quad \widehat{B_n} \quad \widehat{B_{n+1}}}{-a} \; (-) \quad .$$

Thus the rule of denial introduction can be considered as an interpretation of what is sometimes called the "extremal clause" in inductive defintions: "Nothing else is a defining condition of $a$". This sort of inference is called "definitional reflection", as, when carrying it out, we reflect on the definition as a whole. More precisely, we should call it *definitional reflection with respect to denial*, since the original application of this principle is with respect to elimination inferences, which we do not consider here. Obviously, definitional reflection makes derivability non-monotonic with respect to the given definition, i.e., extending the definition may reduce the set of literals derivable.

The denial operation $-$, for which we have just formulated the introduction rule $(-)$, will be called *indirect denial*. This is because its governing rule $(-)$ is *derived* from denial-free clauses for atoms and in this way "parasitic" on them. Indirect denial will be distinguished from direct denial, which is established by individual clauses in (an extended notion of) definition. So "indirect" is another way to express what "definitional reflection" means. It precisely corresponds to the distinction between direct and indirect means of establishing a proposition in Dummett's theory of meaning (see [3]). The direct means are those *encoded in* the definition, whereas the indirect ones are those obtained by *reflection on* the definition. We give two examples for the application of definitional reflection with respect to denial: Given the definition

$$\mathbb{D}_2 \begin{cases} a & \Leftarrow & b \\ a & \Leftarrow & c \end{cases}$$

we obtain

$$-b, -c \vdash_{\mathbb{D}_2} -a,$$

and given the definition

$$\mathbb{D}_3 \begin{cases} a & \Leftarrow & b \\ a & \Leftarrow & c, d \end{cases}$$

we obtain

$$-b, -c \vdash_{\mathbb{D}_2} -a,$$

as well as

$$-b, -d \vdash_{\mathbb{D}_2} -a.$$

So far there are no closed (i.e., assumption-free) proofs using $(-)$, as $(-)$ has only denials as premisses, and we have no way of establishing them (except by denial introduction). This again reflects the indirect character of the denial operation considered. If a proof of the denial of $a$ is called a *refutation* of $a$, we can say, that so far, there may be categorical (= closed = assumption-free) proofs of atoms, but only hypothetical (= open = assumption-dependent) refutations of atoms.

To be fully precise, we should define what definitional reflection means in the case of premiss-free clauses and in the case of atoms for which no defining clause at all is given. In the first case there is simply no rule of definitional reflection defined, i.e. no rule $(-)$ is available. In the second case, $-a$ can be inferred vacuously, as no defining condition (not even the empty one) is available for $a$. However, these limiting cases will not be discussed here in detail.

## 4. Denials in definitional clauses

Suppose we now extend our framework by allowing literals and not only atoms to occur in the bodies of clauses, i.e., we consider clauses of the form

$$a \;\Leftarrow\; \ell_1, \ldots, \ell_n$$

for literals $\ell_1, \ldots, \ell_n$. Let us call such a clause an *assertion clause* (i.e., a clause for the assertion of an atom, i.e., a clause with an atom as its head). Then we can formulate a definition such as

$$\mathbb{D}_4 \begin{cases} a & \Leftarrow & b \\ a & \Leftarrow & -c, d \end{cases}$$

which yields

$$-b, c \vdash_{\mathbb{D}_4} -a \quad \text{and} \quad -b, -d \vdash_{\mathbb{D}_2} -a,$$

as both

$$\frac{-b \quad c}{-a} \quad \text{and} \quad \frac{-b \quad -d}{-a}$$

are instances of (−). This makes it possible in principle to derive a denial without assumptions. Suppose, e.g., the definition

$$\mathbb{D}_5 \begin{cases} a & \Leftarrow & -b \\ b & \Leftarrow & \end{cases}$$

is given. Then the proof

$$\frac{\overline{b}}{-a}\,(-)$$

demonstrates that

$$\vdash_{\mathbb{D}_5} -a$$

holds, i.e., $a$ is refuted outright (= categorically = without assumption).

Therefore, by using the inference rules of definitional closure (introduction of atoms = proof of atoms) and of definitional reflection (introduction of denials of atoms = rejection of atoms), we obtain a calculus for atomic reasoning, in which atoms can be proved or rejected on the basis of assertion clauses.

## 5. Clauses defining denials

The natural next step in extending our framework of definitional reasoning is to admit *denial clauses*, i.e. clauses whose head is a denial. If by that we mean admitting clauses of the form

$$-a \;\Leftarrow\; \ell_1, \ldots, \ell_n,$$

then, in general, by considering clauses of the form

$$\ell \;\Leftarrow\; \ell_1, \ldots, \ell_n,$$

we can dualize the whole framework developed so far. Suppose the literal $\ell$ is defined by the clauses

$$\begin{cases} \ell & \Leftarrow & B_1 \\ & \vdots & \\ \ell & \Leftarrow & B_n \end{cases} .$$

Then we obtain as inferences based on definitional closure:

$$\frac{B_1}{\ell} \quad \ldots \quad \frac{B_n}{\ell}$$

whereas for definitional reflection we obtain

$$\frac{\widehat{B_1} \quad \ldots \quad \widehat{B_n}}{\widehat{\ell}} \; .$$

While the schema of definitional closure now covers introductions for denials, the schema of definitional reflection now covers introductions for assertions (in case $\ell$ is of the form $-a$. For example, given the definition

$$\mathbb{D}_6 \begin{cases} -a & \Leftarrow \quad b \\ -a & \Leftarrow \quad c, -d \\ -b & \Leftarrow \quad e \\ -c & \Leftarrow \quad e \\ d & \Leftarrow \quad e, -c \\ e & \Leftarrow \end{cases}$$

and labelling steps of definitional closure with **c** and those of definitional reflection with **r**, the following proof demonstrates (the assertion of) $a$, with definitional reflection now leading to an assertion claim:

$$\dfrac{\dfrac{\dfrac{}{e}\,\mathbf{c}}{-b}\,\mathbf{c} \qquad \dfrac{\dfrac{}{e}\,\mathbf{c} \quad \dfrac{\dfrac{}{e}\,\mathbf{c}}{-c}\,\mathbf{c}}{d}\,\mathbf{c}}{a}\,\mathbf{r}$$

which would establish

$$\vdash_{\mathbb{D}_6} a.$$

With this proposal we could finish our discussion of assertion and denial in definitional reasoning. We have obtained completely symmetric assertion and denial inferences for atoms, based either on definitional closure or on definitional reflection.

However, not only would we miss out on the square of opposition (which is not a value in itself, of course), but we would also have to retract certain intuitions already developed. One of them was that the rules of definitional closure in a sense represent the *direct* way of establishing a proposition, whereas the rules of definitional reflection are *indirect* ways of doing so. So we distinguished between definitional knowledge stated explicitly in the definition (as the head of a definitional rule) and knowledge only *obtained* from the definition by reflecting on its structure. Putting this in a more traditional terminology, it is the difference between analytic and non-analytic knowledge. Analytic knowledge would be what can be extracted immediately from the definition; it is, so to speak, already there as the head of a clause. In contradistinction to that, non-analytic knowledge results by *interpreting* the definition. It has to be extracted from it in a way that goes beyond just forward reasoning along the definitional clauses. It is not like the "beam in the house" but like the "plant in the seed" to use Frege's ([4], § 88) picture. If we want to keep this fundamental meaning-theoretical distinction, we must terminologically distinguish between a denial which is directly established by means of definitional closure as the head of a definitional clause, and the denial which is established indirectly by definitional reflection, i.e., by means of a non-monotonic inference rule which requires taking the definition as a whole into consideration. In the following we shall write "$\sim a$" for the direct denial of $a$, in contradistinction to "$-a$" for the indirect denial established by definitional reflection. Denial clauses would then be clauses of the form

$$\sim a \;\Leftarrow\; B$$

rather than

$$-a \;\Leftarrow\; B\ .$$

Thus in general, a definitional clause has the form

$$\langle\sim\rangle a \;\Leftarrow\; \langle -\rangle b_1, \ldots, \langle -\rangle b_n$$

where the angle brackets indicate that denial operators may occur at the places indicated.

Now, the operators $\sim$ and $-$, being both denial operations, should not remain unrelated to one another. It is natural to assume that the indirect denial is weaker, or at least not stronger, than the direct denial. So if we have rejected $a$ directly, by inferring the denial of $a$ by definitional closure, this should have at least the power of an indirect rejection, so we should be able to proceed as if with $\sim a$ we also had established $-a$. This leads to assuming a rule of subalternation with respect to denial:

$$\frac{\sim a}{-a}\ (Sub)$$

For example, given the following definition

$$\mathbb{D}_7 \begin{cases} \sim a & \Leftarrow\; b \\ b & \Leftarrow \\ c & \Leftarrow\; -a, -d \\ d & \Leftarrow\; -b \\ d & \Leftarrow\; a, c \end{cases}$$

we can establish

$$\vdash_{\mathbb{D}_7} c$$

by means of the following proof:

$$\dfrac{\dfrac{\dfrac{\overline{b}}{\sim a}}{-a}\,(Sub) \qquad \dfrac{\overline{b} \qquad \dfrac{\dfrac{\overline{b}}{\sim a}}{-a}\,(Sub)}{-d}}{c} \qquad .$$

In this proof, at two places, by use of subalternation the direct denial $\sim a$ of $a$ is turned into the indirect denial $-a$ of $a$ to obtain the premiss $-a$ of the closure rule for $c$ (left application of $(Sub)$) and of the reflection rule for $-d$ (right application of $(Sub)$).

As already indicated at the beginning of this section, there is no reason to restrict definitional reflection to obtain only denial inferences. We can argue that we obtain an indirect sort of assertion from the denial of all possible conditions of directly denying an atom. For that we introduce an additional form of judgment $+a$ to express the *indirect assertion* of $a$, $+a$ being the contradictory opposite (complement) of $\sim a$:

$$\begin{aligned} \widehat{a} &:= -a \\ \widehat{-a} &:= a \\ \widehat{\sim a} &:= +a \\ \widehat{+a} &:= \sim a. \end{aligned}$$

This differs from the concept without the direct/indirect distinction, where we had only a single pair of opposites $(a/{-a})$. Definitional reflection with respect to assertion now leads to the indirect assertion $+a$ of an atom $a$: Given

$$\left\{ \begin{array}{lcl} \sim a & \Leftarrow & B_1 \\ & \vdots & \\ \sim a & \Leftarrow & B_n \end{array} \right.$$

as (direct) denial clauses for $a$, the rule of definitional reflection w.r.t. assertion is

$$\frac{\widehat{B_1} \quad \dots \quad \widehat{B_n}}{+a} \quad .$$

For example, the definition

$$\mathbb{D}_8 \left\{ \begin{array}{lcl} a & \Leftarrow & b \\ a & \Leftarrow & c \\ \sim a & \Leftarrow & d, -e \\ \sim a & \Leftarrow & f \end{array} \right.$$

yields

$$-d, -f \vdash_{\mathbb{D}_8} +a \quad \text{and} \quad e, -f \vdash_{\mathbb{D}_8} +a$$

in addition to

$$b \vdash_{\mathbb{D}_8} a \quad \text{and} \quad c \vdash_{\mathbb{D}_8} a \ .$$

In the bodies of clauses we have admitted assertions of the form $a$ and denials of the form $-a$. From the point of view reached so far, this means that we have admitted direct assertions and indirect denials, which is an obvious asymmetry with respect to assertion and denial. This has, for example, the effect that the indirect $+a$ has no function in proofs, except as an end formula. It can neither occur as a premiss of definitional closure (as it is not admitted to occur in the body of a clause), nor can it occur as a premiss of definitional reflection (as its denial [complement] $\sim a$ cannot occur in the body of a clause either). We remove this asymmetry by allowing for both the direct and the indirect assertions and direct and indirect denials to occur in bodies of clauses. This makes it possible in principle to use any of the possible four forms of judgement as a premiss in an inference of definitional closure and definitional reflection. Furthermore, in analogy with subalternation for denial, we postulate the corresponding rule of subalternation for assertion:

$$\frac{a}{+a}\ (Sub)\ .$$

Even though in the bodies of clauses any form of judgement is allowed, the heads of clauses remain to be either direct assertions or direct denials. Our framework hinges on the distinction between *direct* and *indirect*, and this distinction is drawn by saying that "direct" means "being the head of a program clause".

This leads to the following definition. Let a *direct literal* be either of the form $a$ or of the form $\sim a$ for an atom $a$. Let an *indirect literal* be either of the form $+a$ or $-a$. Let a *literal* be either a direct or an indirect literal. Then a definitional clause has the form

$$\ell \ \Leftarrow\ \ell_1, \ldots, \ell_n$$

where $\ell$ is a direct literal, and $\ell_1, \ldots, \ell_n$ are literals. A definition $\mathbb{D}$ is a list of definitional clauses. Every definitional clause gives rise to a rule of *definitional closure*

$$\frac{\ell_1 \quad \ldots \quad \ell_n}{\ell}$$

whereas for every atom $a$ occurring in the definition, the definition $\mathbb{D}$ as a whole gives rise to a negative and a positive schema of *definitional reflection*

$$\frac{\widehat{B_1} \quad \ldots \quad \widehat{B_m}}{-a} \qquad \frac{\widehat{C_1} \quad \ldots \quad \widehat{C_n}}{+a}$$

provided $B_1, \ldots, B_m$ are the conditions of $a$ and $C_1, \ldots, C_n$ are the conditions of $\sim a$ with respect to the definition $\mathbb{D}$, i.e., the clauses for $a$ and $\sim a$ in $\mathbb{D}$ are as follows:

$$\left\{\begin{array}{lcl} a & \Leftarrow & B_1 \\ & \vdots & \\ a & \Leftarrow & B_m \end{array}\right. \qquad \left\{\begin{array}{lcl} \sim a & \Leftarrow & C_1 \\ & \vdots & \\ \sim a & \Leftarrow & C_n \end{array}\right. .$$

Furthermore, indirect assertions and denials are subalternates to the corresponding direct ones:

$$\frac{a}{+a}\ (Sub) \qquad \frac{\sim a}{-a}\ (Sub)\ .$$

The following is an example showing the interaction of all rules. Note that only the subalternation rules have been labelled as such. In all other cases it is uniquely determined which rule is being applied: Rules of definitional closure only lead to direct literals, rules of definitional reflection only lead to indirect literals.

$$\mathbb{D}_9 \begin{cases} a & \Leftarrow & c, e \\ a & \Leftarrow & \sim d \\ \sim b & \Leftarrow & d \\ c & \Leftarrow & b \\ c & \Leftarrow & \sim d \\ e & \Leftarrow & -a, +d \end{cases}$$

The following proof demonstrates that $d \vdash_{\mathbb{D}_9} e$.

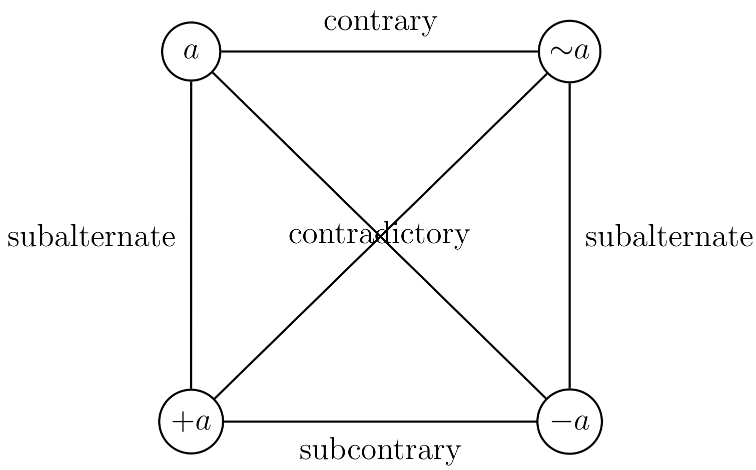
$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{d}{\sim b}}{-b}\,(Sub) \quad \cfrac{d}{+d}\,(Sub)}{-c} \quad \cfrac{d}{+d}\,(Sub)}{-a} \quad \cfrac{d}{+d}\,(Sub)}{e}$$

## 6. The square of opposition

We have now obtained a structure which fits into the square of opposition. The direct forms of judgement $a$ and $\sim a$ are to be placed in the $A$ and $E$ nodes, and the indirect forms of judgement $+a$ and $-a$ in the $I$ and $O$ nodes, respectively. Due to our subalternation rules $(Sub)$ and the complementation operation $\widehat{\ }$ with the properties

$$\begin{array}{rcl} \widehat{a} & := & -a \\ \widehat{-a} & := & a \\ \widehat{\sim a} & := & +a \\ \widehat{+a} & := & \sim a. \end{array}$$

we have already interpreted the right and left sides (subalternation) and the diagonals (contradictoriness) of the square. It remains to give a meaning to contrariety and subcontrariety.




Contrariety is the opposition between direct judgements $a$ and $\sim a$, i.e. between direct assertion and direct denial. If we want to relate it to the square, we have to say what it means in our framework that contrary judgements cannot both be true (but may both be false). In our proof-theoretic context, the most natural way is to interpret truth as provability and falsity as the provability of the denial (complement). Now it is easy to present a definition in which both $a$ and $\sim a$ are provable, the most trivial one being

$$\mathbb{D}_{10} \left\{ \begin{array}{rl} a & \Leftarrow \\ \sim a & \Leftarrow \end{array} \right.$$

So there is no way of claiming that contrary opposites *cannot* be proved.

Therefore we propose a change of perspective. Rather than claiming that the square depicts deductive relationships naturally enforced by our inference rules, we use it as a structure to classify definitions. Some definitions may obey principles represented in the square, others may not. We might, e.g., say that a definition *respects* contrariety, because it is consistent, and so on. In this sense, the square is read as specifying certain deductive features of a given definitions. We do not argue here that these specifications have to be be fulfilled.

Thus we propose the following (metalinguistic) definition:

*Definition 1. A definition* $\mathbb{D}$ *respects*

*(1)* contrariety*, if for no atom* $a$*, both* $\vdash_{\mathbb{D}} a$ *and* $\vdash_{\mathbb{D}} \sim a$*,*
*(2)* subcontrariety*, if for no atom* $a$*, both* $\vdash_{\mathbb{D}} \widehat{+a}$ *and* $\vdash_{\mathbb{D}} \widehat{-a}$
*(3)* weak contradictoriness*, if for no atom* $a$*, both elements of a complementary pair* $(a/-a)$ *or* $(\sim a/+a)$ *are provable*
*(4)* strong contradictoriness*, if for every atom* $a$*, exactly one element of each complementary pair* $(a/-a)$ *and* $(\sim a/+a)$ *is provable.*

$\mathbb{D}$ weakly *respects the* square of opposition*, if it respects (1)–(3). It* fully *respects the* square of opposition*, if it respects (1)–(4).*

Obviously, respecting contrariety and respecting subcontrariety means the same. This is something we are used to from the traditional square of opposition. To say that $SaP$ and $SeP$ are not both true is the same as saying that their negations $\widehat{SaP}$ and $\widehat{SeP}$, i.e., $SoP$ and $SiP$, are not both false. This is nothing but the duality of true and false with respect to negation (assuming, of course, that $SoP$ and $SiP$ are indeed the negations of $SaP$ and $SeP$, respectively).

The distinction between weak and strong contradictoriness is due to the fact that in our proof-theoretic framework the provability of a literal $\ell$ and the provability of its complement $\widehat{\ell}$ do not necessarily exhaust all possibilities, as neither $\ell$ nor $\widehat{\ell}$ may be provable in the given definition. Respecting strong contradictoriness corresponds to some sort of completeness and is therefore a very strong condition.

An example demonstrating that in (1) both $a$ and $\sim a$ can be false, which in our context is expressed by $\vdash_{\mathbb{D}} \widehat{a}$ and $\vdash_{\mathbb{D}} \widehat{\sim a}$, would be given by an $a$ which is not defined in $\mathbb{D}$, i.e., for which there is no clause in $\mathbb{D}$ with $a$ od $\sim a$ as its head. In this case, $\vdash_{\mathbb{D}} -a$, which is the same as $\vdash_{\mathbb{D}} \widehat{a}$, and $\vdash_{\mathbb{D}} +a$, which is the same as $\vdash_{\mathbb{D}} \widehat{\sim a}$, hold vacuously. At the same time, this is an example demonstrating that in (2) both $+a$ and $-a$ can be true, which in our context is expressed by $\vdash_{\mathbb{D}} +a$ and $\vdash_{\mathbb{D}} -a$.

In the previous section, we took the position that indirect judgements should be intrinsically related to their direct forms by means of the rules of subalternation. However, from the specification perspective taken now, one might consider giving up postulating ($Sub$) as built-in rules. One might instead argue that the rules of definitional closure and of definitional reflection are the only genuine rules putting a definition into action, and in this sense are sufficient to explicate the meaning of direct and indirect assertion and denial. Subalternation would then be a non-primitive inference rule that may (but need not) be admissible on the basis of the closure and reflection rules. This means that the admissibility of subalternation would not always be trivial, but a matter of (mathematical) fact. If we choose this option, we have have to add a clause to the above definition yielding the following Definition.

*Definition 2. A definition* $\mathbb{D}$ *respects*

*(1)* contrariety*, if for no atom a, both* $\vdash_{\mathbb{D}} a$ *and* $\vdash_{\mathbb{D}} \sim a$*,*
*(2)* subcontrariety*, if for no atom a, both* $\vdash_{\mathbb{D}} \widehat{+a}$ *and* $\vdash_{\mathbb{D}} \widehat{-a}$
*(3)* subalternation*, if the rules* ($Sub$) *are admissible with respect to* $\mathbb{D}$
*(4)* weak contradictoriness*, if for no atom a, both elements of a complementary pair* $(a/-a)$ *or* $(\sim a/+a)$ *are provable*

(5) *strong contradictoriness, if for every atom a, exactly one element of each complementary pair* $(a/{-a})$ *and* $({\sim a}/{+a})$ *is provable.*

$\mathbb{D}$ weakly *respects the* square of opposition*, if it respects (1)–(4). It* fully *respects the* square of opposition*, if it respects (1)–(5).*

Whether to consider this option or not must be left open here. In the framework without subalternation as a primitive rule, we can show the following:

*Theorem 1. In the framework without subalternation as a primitive rule, every definition* $\mathbb{D}$ *respects weak contradictoriness.*

*Demonstration. We proceed by induction on the length of proofs. Suppose* $a$ *and* $-a$ *have both been proved. Then the last step of the proof of* $a$ *has the form*

$$\frac{\ell_1 \ \dots \ \ell_n}{a} \, ,$$

*where*

$$a \ \Leftarrow \ \ell_1, \dots, \ell_n$$

*is a definitional clause of* $a$*, whereas the last step of the proof of* $-a$ *has the form*

$$\frac{\dots \ \ \widehat{\ell_i} \ \ \dots}{-a}$$

*for some* $i$*. This means that there are proofs of* $\ell_i$ *and* $\widehat{\ell_i}$ *of shorter length.*

This result is an argument for the framework without subalternation, as it guarantees a property one should expect to hold in any case. With subalternation as a primitive rule, we might be able to prove both $a$ and $-a$ (e.g., in the case of $\mathbb{D}_{10}$) and thus would destroy the idea of the complementary relationship expressed by the operation $\widehat{\ \ }$.

As a corollary of this theorem we immediately obtain:

*Corollary. If* $\mathbb{D}$ *respects subalternation, it respects contrariety.*

This result is also quite significant. It relates the consistency of a definition to the relation between direct and indirect assertion/denial. Only in a consistent definition we can infer the indirect from the direct forms.

If subalternation is presupposed as a primitive rule, we only have the following: result.

*Theorem 2. In the framework with subalternation as a primitive rule, every definition* $\mathbb{D}$ *which respects contrariety respects weak contradictoriness.*

*Demonstration. Given proofs of* $a$ *and of* $-a$*, the second proof cannot be obtained by subalternation (since otherwise we would have proofs of* $a$ *and of* $\sim a$*.) This means that we can argue as in the demonstration of Theorem 1.*

The following is an example of a definition, which fully respects the square of opposition.

$$\mathbb{D}_{11}\left\{\begin{array}{rcl} a & \Leftarrow & b, c \\ \sim a & \Leftarrow & -b \\ \sim a & \Leftarrow & -c \\ b & \Leftarrow & \\ \sim b & \Leftarrow & -b \\ \sim c & \Leftarrow & \\ c & \Leftarrow & +c \end{array}\right.$$

As can easily be checked, the literals $\sim a, -a, b, +b, \sim c, -c$ are provable, whereas $a, +a, \sim b, -b, c, +c$ are unprovable in $\mathbb{D}_{11}$.

## 7. Discussion

We presented a framework for definitional reasoning, in which two denials and two assertions are distinguished. In contradistinction to approaches that introduce some strong form of negation into logical reasoning such as Nelson [8]), we also extended the notion of assertion and received full symmetry between assertion and denial. This symmetry led to our square of opposition. As there is a weak ("indirect") denial opposed to strong ("direct") assertion, there is a weak ("indirect") assertion opposed to strong ("direct") denial. The contradictory opposite to the strong concept is the opposite weak concept. The strong concepts mean that we can establish the proposition in question by means of the given clauses directly, applying in the last step an inference following the *direction* of the clause (thus "direct"). The weak concepts mean that we can establish a proposition by showing that the premisses of its opposite can be rejected. Here "rejection" does not mean *failure to prove*, but *proving the contradictory opposite* of at least one premiss. This reasoning is *indirect* as we have to use definitional reflection, i.e., we have to reflect on the definition as a whole. This represents a novel sense of duality which is different from treatments of weak and strong negation.

The notions of "direct" and "indirect" assertion and denial appeal to intuitions which are related to the idea of direct vs. indirect proofs in Dummett-Prawitz-style proof-theoretic semantics. However, there they are used to describe the relationship between introduction and elimination rules and not between assertion and denial. Moreover, we are here dealing with reasoning on the basis of arbitrary atomic definitions, i.e., with definitional reasoning in general. Nonetheless, in spite of these differences, there is some underlying general duality at work which manifests itself in various special forms like "assertion" vs. "assumption", "assumption" vs. "denial", "conditions" vs. "consequences", "introduction" vs. "elimination", "right" vs. "left" [sequent calculus] — a duality, which needs to be further elaborated (see [5, 6]).

In order to gain a clear picture, we have exclusively dealt with the relation between assertion and denial, keeping aside elimination inferences as well as proofs from assumptions. So what we have sketched here, was only a first step. Our approach can be extended in various directions.

(1) We could combine the duality of assertions and assumptions (or of introduction and elimination rules) with the duality of assertions and denials (or of direct vs. indirect introduction rules), and investigate how these notions interact. This opens a new field of interesting observations with new harmony principles for the classification of definitions. As elimination rules bring with them the "standard" (normally intuitionistic) notion of negation, we may examine how direct denial relates to (and perhaps coincides with) standard negation.

(2) We might investigate what happens if we allow for implications to occur in the bodies of clauses. So far, we have only considered literals divided by commas as bodies. Including implication is not trivial at all, as one enters the field of non-monotonic inductive definitions. Furthermore, there are several options of denying implication, where

$$\widehat{a \to b} \;:=\; a, \widehat{b}$$

is only one possibility among others. Pursuing this idea would have to take ideas from connexive logic into account (see [13]).

(3) For real applications we must leave the area of propositional reasoning behind and consider clauses with first-order atoms containing individual variables. In that case defining definitional reflection for denial and assertion will require considerations quite analogous to inversion principles in standard proof-theoretic semantics. There are several options to formulate them, all of which require a careful handling of variables and substitutions (see [12]).

(4) One might develop logic programming paradigms, in which logic programs have the form of definitions with assertions and denials, and define a way to compute bindings (substitutions). This would lead into the area of extended logic programming where one considers negated heads in program clauses (see [2]).

## Conclusion

It is fascinating to see how such an elementary figure as the square of opposition can exhibit a structure relevant to an area that at first sight looks quite remote to it. I should like to thank Jean-Yves Béziau for asking me to contribute a paper to this congress and for encouraging me to think about how to relate certain aspects of my work to the square of opposition. Special thanks are due to Wagner de Campos Sanz for numerous helpful comments on a draft of this paper, and for many stimulating discussions on its topic. I also thank two anonymous reviewers for various substantial comments and suggestions.

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